UNIVERSITY
OF ZIELONA GÓRA

DOCTORAL DISSERTATION

FIELD: NATURAL SCIENCES
DISCIPLINE: MATHEMATICS

On continuous solutions of linear functional equations of infinite order

(*O rozwiązaniach ciagłych liniowych równań funkcyjnych nieskończonego rzędu*)

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I accept the presented text of the PhD thesis

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(signature of the supervisor)

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Zielona Góra, 2020

Dedicated to Wiktoria Martyńska
Had it not been for your support, I would have never finished my work.

Preface

The presented dissertation is devoted to linear functional equations of inifnite order in a single variable. Informally we can say that this is a kind of functional equations in which infinitely many terms appear. Therefore we very often use a series (if there are countable many terms) or an integral (if uncountable many terms appear) to write down such an equation. Since we can always view on a series as on a Lebesgue integral with respect to a discrete measure, we see that a shape of these equations is detemined by the measure. In truth, in this text we shall meet probability measures mostly.

In this PhD thesis we provide some criterions imposed on measures and functions appearing in functional equations under which we can find a solution in the class of bounded continuous functions. The results presented in this dissertation can be obviously applied to equations of finite order.

My research led in Chapter 2 and 3 concentrates on solutions attaining its global extremum. I analyse also what happened when such a value is not reached by the solution. The second area of my work, presented in Chapter 4, is an examination of an influence of the so-called invariant compact sets on solutions. It turns out that they play the crucial role in the theory of linear functional equations in a single variable.

Contents

Introduction

The scientific research presented in this PhD thesis is devoted to linear functional equations of infinite order. The starting point of my work was a problem posed by Gregory Derfel during the *21st European Conference on Iteration Theory* held in Innsbruck (Austria) in 2016. He asked if there existed a non-constant bounded continuous solution $\varphi : \mathbb{R} \to \mathbb{R}$ of the equation

$$\varphi(x) = \frac{1}{2}\varphi(x-1) + \frac{1}{2}\varphi(-2x). \tag{0.1}$$

The question was repeated by him one year later on the *55th International Symposium on Functional Equations* in Chengdu (China).

The above problem has been formulated in a very simple way, equation (0.1) has an uncomplicated shape. We need no advanced mathematical notions to pose the above question – it can be well understood by students of the first year of technical studies or even students of high school with an extended mathematics programme. Answering to the presented question turned out to be very difficult, as often as it is the case with mathematical problems which can be formulated very easy. The reasons of this fact and a partial answer to the Derfel's question are presented in this dissertation.

The problem posed by Derfel is tightly connected with the so-called *archetypal equation*, i.e. a functional equation of the form

$$\varphi(x) = \iint_{\mathbb{R}^2} \varphi\big(a(x-b)\big)\mu(da, db), \tag{0.2}$$

where μ is a given Borel probability measure defined on $\mathbb{R}^2$ and $\varphi : \mathbb{R} \to \mathbb{R}$ is a bounded continuous function. As we know, if equation depends on a parameter, then its shape can be various for different values of this parameter. In the case like this a role of the parameter is played by the measure μ and it can change the form of equation (0.2) drastically. As we will see in this text, the archetypal equation may sometimes reduce even to differential equations.

Equation (0.2) is very well examined in the case $\mu\big((0,\infty)\times\mathbb{R}\big)=1$. In the paper [13] from 1989 Gregory Derfel discovered that a behaviour of the set of bounded continuous solutions of the archetypal equation depends significantly on the value of the number

$$K := \iint_{\mathbb{R}^2} \ln|a|\mu(da,db).$$

He proved that, under some technical assumptions, the archetypal equation has not non-constant bounded continuous solutions provided $K<0$. He constructed also a non-constant bounded continuous solution for $K>0$ when $\mu\big((0,\infty)\times\mathbb{R}\big)=1$ additionaly. The fact that the support of μ is contained in the right half-plane is crucial in his proof. The case when $K>0$ and $\mu\big((-\infty,0)\times\mathbb{R}\big)>0$ has not been explored very well today and equation (0.1) examplifies this situation. In particular, in the thesis I would like to examine this class of functional equations and the topics which can help to find their solutions.

The whole text is splitted into five chapters. The first chapter contains basic facts from different branches of mathematics, mainly measure theory and dynamical systems. In particular, in Chapter 1 we present notations used in this text and basic terms which will be applied in sequent parts of the dissertation. The theorems contained in this part are presented without proofs with one exception but references are always given.

Chapter 2 is devoted to the archetypal equation. At the beginning of this part we present some results coming from the papers [10] and [11] by L. Bogachev, G. Derfel and S. Molchanov. They can help to understand a motivation of Gregory Derfel to proposed equation (0.1). The further sections of this chapter are based on the articles [36] and [37] which concern solutions of the archetypal equation attaining the global extremum. In particular, in Section 2.3 we will prove that each bounded continuous solution of equation (0.2) which attains its global extremum must be constant in the case $\mu((-\infty,0)\times\mathbb{R})>0$ if the measure μ satisfies some additional technical assumptions (see details in Theorem 2.3.1 and Theorem 2.3.2). We add that Theorem 2.3.1 implies that every bounded continuous solution of equation (0.1) is constant if it attains the global extremum. Section 2.4 contains a discussion on properties of solutions which do not attain the global extremum. In this section we prove a theorem saying that if such a non-constant solution exists, then it must be oscilating at the inifnities. It is worth adding that the existence of non-constant solutions in the class of bounded continuous functions is still an open problem.

Chapter 3 is a consequence of attempts in generalizing the results from Sections 2.3 and 2.4 on linear functional equations for which we have non-affine transforms of arguments. More precisely, the cosniderations of Chapter 3 refers to functional equations of the form

$$\varphi(x) = \sum_{i \in I} p_i \varphi\big(f_i(x)\big), \tag{0.3}$$

where $I \subseteq \mathbb{Z}$ is fixed, $p_i \in (0,1)$ are summing up to 1 and $f_i : \mathbb{R} \to \mathbb{R}$ is a homeomorphism for every $i \in I$. Section 3.1 contains a conception of the *compatibility conditions* which improve some dynamical aspects of iterates of the functions $f_i, i \in I$. In Section 3.2 we examine equation (0.3) under the comptability conditions. In particular, in this part we show that for a family $\{f_i\}_{i \in I}$ fulfilling any compatibility condition every bounded continuous solution $\varphi : \mathbb{R} \to \mathbb{R}$ of equation (0.3) is constant provided φ attains the global extremum. Section 3.3 contains the asymptotical analysis of solutions in the case when $\{f_i\}_{i \in I}$ contains at least one decresaing function. An example demonstrating the results of this chapter appears at the end of this section.

Chapter 4 is devoted to the most general linear functional equation appearing in this text. In this part we have given a probability space $(\Omega, \mathcal{A}, P)$, a complete metric space (X, d), a separable Banach space $(Y, \|\cdot\|)$ and a function $f : \Omega \times X \to X$ such that $f(\cdot, x)$ is $\mathcal{A}$-measurable for every fixed $x \in X$. There we are interested in bounded Borel solutions $\varphi : X \to Y$ of the equation

$$\varphi(x) = \int_{\Omega} \varphi\big(f(\omega, x)\big) P(d\omega), \tag{0.4}$$

where the above integral denotes the Bochner integral. Equations of this form were examined by several authors for years. For example, more than 100 years ago Wacław Sierpiński considered in [34] a particular case of the above equation to characterize Cantor functions. One can observe that the archetypal equation is a very particular case of (0.4). It is worth adding that equations with affine transforms of arguments play a prominent role in applications. The reader is refered to [32], [33] or [35] for instance. Nowadays many papers connected with equation (0.4) and its generalization are written by a group of Polish mathematicans, namely: K. Baron, R. Kapica and J. Morawiec. They studied the various types of the convergence of the inner functions. The paper [3], written by Karol Baron, examplified their methods very well. The articles [5] and [6] are connected with a more general equation than (0.4). In the mentioned papers K. Baron and J. Morawiec examined the Lipschitzian solutions. Furthermore, to get solutions of equation (0.4) they also applied regular Markov-Feller operators which map the space of all Borel probability measures into itself (see [24] for instance). J. Jarczyk and W. Jarczyk are also interested in equation (0.4). In the recently published article [21] they considered equation (0.4) when the inner functions are means. Moreover, W. Jarczyk studied intensively linear functional equations under commutativity conditions – see [22]. We announce that some commutativity conditions will be assume in whole Chapter 4. General surveys of functional and functional-differential equations are found in G. Derfel [14], K. Baron and W. Jarczyk [4] and K. Baron [2].

Chapter 4 is based principally on the article [38]. The main topic of this part are invariant comptact sets. We say that a set $K \subseteq X$ is invariant if

$$\bigcup_{x \in K} \{f(\omega, x)\} \subseteq K \quad \text{for all } \omega \in \Omega.$$

In Chapter 4 we will see that an influence of these sets on solutions of linear functional equations is crucial. Section 4.1 contains the main result and its proof. In Section 4.2 we prove some theorems connected with the existence of invariant compact sets. In Theorem 4.1.1 we impose some technical assumptions, that is why in Section 4.3 we present a lot of particular cases of this theorem. Section 4.4 is devoted to the case when the order of equation (0.4) is finite. The last section contains some examples and applications of invariant compact sets to linear functional equations.

In the last chapter we are going back to equation (0.1). In Section 5.1 we prove, among others, that for this equation there are no non-empty invariant compact sets. This is one of the reasons why the equation proposed by Derfel is so hard to solve. The other reasons are disccused also therein. In the next, and last, part of this dissertation we pose some open problems.

Chapter 1

Preliminaries

1.1 Dynamical systems

At the beginning we will introduce a relation in the set of reals which will be useful in our further analyses. We will call it as a *Kuratowski relation* although this term was known before him – in fact, Kazimierz Kuratowski discovered a generalization of the below relation for noninvertible maps.

Definition 1.1.1. Let $I \subseteq \mathbb{Z}$ be fixed and assume that we have a family $\{f_i\}_{i \in I}$ of homeomorphisms of the real line $\mathbb{R}$. We say that $x, y \in \mathbb{R}$ are in the relation $\sim$, and write $x \sim y$, if there exist $k \in \mathbb{N}$, $i_1, ..., i_k \in I$ and $\varepsilon_1, ..., \varepsilon_k \in \{-1, 1\}$ such that

$$y = \left(f_{i_1}^{\varepsilon_1} \circ ... \circ f_{i_k}^{\varepsilon_k}\right)(x).$$

One can simply prove the following fact.

Remark 1.1.1. The relation $\sim$ is an equivalence relation in the set $\mathbb{R}$. For each $x \in \mathbb{R}$ the equivalence class of x will be denoted by $[x]_\sim$.

We will meet the above equivalence relation in Chapters 3 and 5. It turns out that this relation is very helpful and appear naturally in the theory of functional equations in a single variable.

Remark 1.1.2. It is worth adding that composition of functions will be used many times in the thesis. We will use the symbol f^n, where f denotes a transform of an arbitrary set X into itself and $n \in \mathbb{N}$, to represent the n-th iterate of the function f.

Below we present a generalization of the notion of contraction. Firstly we will recall a definition of classical contractions.

Definition 1.1.2. Let (X, d_1) and (Y, d_2) be metric spaces and $f : X \to Y$ be a function. We say that f fulfills the Lipschitz condition with constant $L \in (0, \infty)$ if the inequality

$$d_2\big(f(x), f(y)\big) \le L d_1(x, y)$$

is satisfied for all $x, y \in X$. If L is less than 1, then we say that f is a *contraction*.

We are going to present a wider class than that of classical contractions. Before it happens we have to introduce a useful type of functions. They are indispensable in a definition of generalized contractions.

Definition 1.1.3 (J. Matkowski, [27])**.** We say that a non-negative function $g : [0, \infty) \to \mathbb{R}$ is a *comparison function* if g is non-decreasing and the sequence of its iterates $\big(g^n(t)\big)_{n\in\mathbb{N}}$ converges to 0 for every $t \in [0, \infty)$.

Remark 1.1.3. It is easy to see that any comparison function may be discontinuous in general. Moreover, one can check that $g(0) = 0$ and $g(t) < t$ for all positive values of t.

Now we are in a position to define generalized contractions. The next definition comes from Janusz Matkowski.

Definition 1.1.4 (J. Matkowski, [27])**.** Let (X, d_1) and (Y, d_2) be metric spaces and $g : [0, \infty) \to [0, \infty)$ be a comparison function. We say that $f : X \to Y$ is a *Matkowski type contraction* (with a comparison function g) if the inequality

$$d_2\big(f(x), f(y)\big) \le g\big(d_1(x, y)\big)$$

is satisfied for all $x, y \in X$.

Remark 1.1.4. We can observe that Matkowski type contractions are continuous despite the fact that g may have discontinuities. Furthermore, every classical contraction is a Matkowski type contraction. More precisely, if (X, d_1) and (Y, d_2) are metric spaces and $f : X \to Y$ is a contraction with a Lipschitz constant L, then it is a Matkowski type contraction with a comparison function g given by the formula $g(t) = Lt$.

The next example shows that the class introduced by Janusz Matkowski is indeed bigger than that of classical contractions.

Example 1.1.1. Consider the interval $I = [0,1]$ equipped with the metric generated by the absolute value norm $|\cdot|$. Let $f : I \to I$ be defined by

$$f(x) = \frac{x}{1+x}.$$

One can note that f is continuously differentiable and

$$f'(x) = \frac{1}{(1+x)^2} \quad \text{for every } x \in [0,1].$$

Hence f fulfills the Lipschitz condition with the constant $L = 1$ and, since $f'(0) = 1$, this is the optimal choice and, consequently, f is not a Banach contraction. But it is a Matkowski type contraction with a comparison function $g = f$, as for all $x, y \in [0,1]$ we have

$$|f(x) - f(y)| = \left|\frac{x}{1+x} - \frac{y}{1+y}\right| = \left|\frac{x-y}{(1+x)(1+y)}\right| = \frac{|x-y|}{1+x+y+xy}$$
$$\leq \frac{|x-y|}{1+|x-y|} = g\big(|x-y|\big).$$

Matkowski type contractions share many dynamical properties with classical contractions. The next theorem generalizes the well known Banach contraction principle (see [1]).

Theorem 1.1.1 (Matkowski fixed-point theorem, [27], also [17])**.** *Let (X,d) be a complete metric space. If $f : X \to X$ is a Matkowski type contraction, then f has a unique fixed point $\xi \in X$ and it is globally attractive, i.e. for every $x \in X$ the sequence $(f^n(x))_{n\in\mathbb{N}}$ converges to ξ.*

In Chapter 4 we are working with compact subsets of metric spaces. It is clear that we can measure a distance between any two compact sets in many ways but there is one special method due to Hausdorff.

Definition 1.1.5. Let (X,d) be a metric space and K, C be its compact subsets. We define their Hausdorff distance $d_H(K,C)$ by

$$d_H(K,C) := \inf\big\{\varepsilon \in (0,\infty) : \quad K \subseteq C(\varepsilon) \text{ and } C \subseteq K(\varepsilon)\big\},$$

where

$$A(\varepsilon) := \bigcup_{x\in A} \{y \in X : \quad d(x,y) < \varepsilon\} \quad \text{for every } A \subseteq X \text{ and } \varepsilon \in (0,\infty).$$

Remark 1.1.5. One can check that d_H is a metric in the space of all compact subsets of X. Then the function d_H is called the *Hausdorff metric*. It is worth adding that the space of all compact subsets of a complete metric space (X, d) becomes a complete metric space with the Hausdorff metric. We will denote this space by $(\mathcal{K}(X), d_H)$.

In the theory of dynamical systems iterated function systems play a fundamental role. We remind their definition.

Definition 1.1.6. Let (X, d) be a metric space and $N \in \mathbb{N}$. We say that $\{f_i : X \to X| \quad i = 1, 2, ..., N\}$ is an *iterated function system* (or IFS for short) if f_i is a contraction for every $i = 1, 2, ..., N$.

In [20] Hutchinson showed that for iterated function systems defined on a complete metric space X there exists a special compact set, called an attractor, which can be treated as a counterpart of a global attractive fixed point for an individual function. In fact, it is a global attractive fixed point of a set-valued contraction mapping the space $(\mathcal{K}(X), d_H)$ into itself. For the details and the proof the reader is refered to [20].

In the same way as we defined generalized contractions, we introduce generalized iterated function systems. We accept the following definition.

Definition 1.1.7. Let (X, d) be a metric space and $N \in \mathbb{N}$. We say that $\{f_i : X \to X| \quad i = 1, 2, ..., N\}$ is a *generalized iterated function system* (or GIFS for short) if f_i is a Matkowski type contraction for every $i = 1, 2, ..., N$.

Theorem 1.1.2 (P. Jaros, Ł. Maślanka, F. Strobin, [23]). *Let (X, d) be a complete metric space and assume that a family $\{f_i : X \to X| \ i = 1, 2, ..., N\}$, where $N \in \mathbb{N}$ is given, is a GIFS. Then there exists a unique compact set $K \subseteq X$ such that*

$$\bigcup_{i=1}^{N} f_i(K) = K. \tag{1.1}$$

Moreover, if $C \subseteq X$ is an arbitrary compact set, then the sequence $(C_n)_{n\in\mathbb{N}}$, defined by $C_1 = C$ and

$$C_{n+1} = \bigcup_{i=1}^{N} f_i(C_n) \quad \text{for every } n \in \mathbb{N},$$

is convergent (with respect to the Hausdorff metric) to the attractor K.

The above theorem is an extension of the original Hutchinson Theorem from IFS on GIFS.

In Chapter 4 we will see that every attractor is, in particular, an invariant compact set with respect to a GIFS. Other properties of attractors are not used in this dissertation.

1.2 Measure and integral

In this dissertation we will be working only with finite measures, but in fact they will be probability measures mostly. We recall some definitions and basic facts from the measure theory and fix the notation. We start with the following

Definition 1.2.1. Let $\mathcal{A}$ be a family of subsets of the nonempty set Ω. We say that this family is a *σ-algebra* (of subsets of Ω) if

(i) $\mathcal{A}$ is nonempty,

(ii) for every set A from $\mathcal{A}$ its complement $\Omega \setminus A$ belongs to $\mathcal{A}$,

(iii) for each sequence $(A_n)_{n\in\mathbb{N}}$ of sets from $\mathcal{A}$ the sum $\bigcup_{n=1}^{\infty} A_n$ belongs to $\mathcal{A}$.

The pair $(\Omega, \mathcal{A})$ is called a *measurable space* and elements of $\mathcal{A}$ are called *measurable sets.*

Definition 1.2.2. Let $(\Omega, \mathcal{A})$ be a measurable space and let $\mu : \mathcal{A} \to [0, \infty]$ be a function. We say that μ is a *measure* if $\mu(\emptyset) = 0$ and for all sequences $(A_n)_{n\in\mathbb{N}}$ of pairwise disjoint sets from $\mathcal{A}$ we have

$$\mu\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} \mu(A_n).$$

The triple $(\Omega, \mathcal{A}, \mu)$ is called a *measure space.* If $\mu(\Omega) < \infty$, then we say that the measure μ is finite or equivalently the space $(\Omega, \mathcal{A}, \mu)$ is finite. Furthermore we say that μ is a *probability measure* if it is normed, i.e. $\mu(\Omega) = 1$, and then the triple $(\Omega, \mathcal{A}, \mu)$ is called a *probability space.*

If Ω is a topological space, then we can define a σ-algebra generated by its open subsets. We denote it by $\mathcal{B}(\Omega)$ and call its members *Borel sets.* Any measure on the space $\big(\Omega, \mathcal{B}(\Omega)\big)$ is said to be a *Borel measure.*

For Borel measures it is possible to define a set called a *support of the measure.* In fact, supports of all measures considered in the present thesis are subsets of the Euclidean space $\mathbb{R}^N$, where $N \in \mathbb{N}$. We accept the following definition.

Definition 1.2.3. Let $N \in \mathbb{N}$ be given and $\big(\mathbb{R}^N, \mathcal{B}(\mathbb{R}^N), \mu\big)$ be a measure space. A set of all $x \in \mathbb{R}^N$ such that each neighbourhood of x is of positive measure μ is called the *support of the measure* μ and is denoted by suppμ.

Remark 1.2.1. One can note that the support of any Borel measure is a closed set of space, and hence it is a Borel set of full measure.

If we have a finite family of measurable spaces, then we can construct more complex spaces using the Cartesian product.

Definition 1.2.4. Let $k \in \mathbb{N}$ and let $(\Omega_1, \mathcal{A}_1), \ldots, (\Omega_k, \mathcal{A}_k)$ be measurable spaces. The σ-algebra generated by sets of the form $A_1 \times \ldots \times A_k$, where A_i belongs to $\mathcal{A}_i$ for each $i = 1, 2, ..., k$, is denoted by $\mathcal{A}_1 \otimes \ldots \otimes \mathcal{A}_k$. The measurable space $\big(\Omega_1 \times \ldots \times \Omega_k, \mathcal{A}_1 \otimes \ldots \otimes \mathcal{A}_k\big)$ is called a *product space.*

If all spaces are equipped with the finite measures, then we can construct a special measure on the product space. The details are provided by the following result.

Theorem 1.2.1 (Theorem 18.2, [7])**.** *Let $(\Omega_1, \mathcal{A}_1, \mu_1)$ and $(\Omega_2, \mathcal{A}_2, \mu_2)$ be finite measure spaces. Then there exists a unique measure μ defined on $\mathcal{A}_1 \otimes \mathcal{A}_2$ satisfying the equality*

$$\mu(A \times B) = \mu_1(A) \cdot \mu_2(B)$$

for every $A \in \mathcal{A}_1$ and $B \in \mathcal{A}_2$.

Remark 1.2.2. It is easy to see that we can inductively extend the above theorem to the case of any finite number of measure spaces. Extension to products of infinite many spaces (in fact they have to be probability spaces) can be found in Theorem 3.5.1 in [9] for example. However, we shall use only finite products of measure spaces in the presented text.

Remark 1.2.3. A measure defined on the product space appearing in the above theorem is called a *product measure* and is denoted by $\mu_1 \otimes \mu_2$. If we take k copies of the space $(\Omega, \mathcal{A}, \mu)$, where $k \in \mathbb{N}$, then we denote the product space of this k copies by $(\Omega^k, \mathcal{A}^k, \mu^k)$ for simplicity.

It is a good moment to remind the famous Tonelli-Fubini Theorem which will be applied a few times in the next chapters.

Theorem 1.2.2 (Theorem 4.4.5, [16]). *Let* $(\Omega_1, \mathcal{A}_1, \mu_1)$ *and* $(\Omega_2, \mathcal{A}_2, \mu_2)$ *be finite measure spaces and* $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$ *be an* $\mathcal{A}_1 \otimes \mathcal{A}_2$*-measurable function. Assume that* f *is non-negative or* f *is integrable with respect to the measure* $\mu_1 \otimes \mu_2$*. Then*

$$\int\limits_{\Omega_1\times\Omega_2} f d(\mu_1 \otimes \mu_2) = \int\limits_{\Omega_1} \left[\int\limits_{\Omega_2} f(x,y)\mu_2(dy) \right] \mu_1(dx) = \int\limits_{\Omega_2} \left[\int\limits_{\Omega_1} f(x,y)\mu_1(dx) \right] \mu_2(dy).$$

Now we are going to a theorem allowing to change the measure in the Lebesgue integral. Firstly, we will recall how to generate new measures on a given probability space.

Theorem 1.2.3 (Theorem 16.9, [7]). *Let* $(\Omega, \mathcal{A}, \mu)$ *be a probability space and* $g : \Omega \to \mathbb{R}$ *a non-negative* $\mathcal{A}$*-measurable function such that* $\int_\Omega g d\mu = 1$*. Then a function* $\nu : \mathcal{A} \to [0, 1]$*, given by the formula*

$$\nu(A) = \int_A g d\mu \quad \text{for every } A \in \mathcal{A},$$

is a probability measure on the space $(\Omega, \mathcal{A})$*. Furthermore* $\nu(A) = 0$ *whenever* $\mu(A) = 0$ *for every* $A \in \mathcal{A}$*.*

The function g from the above theorem is called a *density* of the measure ν with respect to μ. Sometimes it is denoted by $\frac{d\nu}{d\mu}$ and called a *Radon-Nikodym derivative* of ν with respect to μ.

Let us recall a relation occuring sometimes between two measures defined on the same measurable space.

Definition 1.2.5. Let $(\Omega, \mathcal{A})$ be a measurable space and let μ and ν be measures on $(\Omega, \mathcal{A})$. We say that ν is *absolutely continuous* with respect to μ and denote this fact by $\nu \ll \mu$ if the implication

$$\mu(A) = 0 \Longrightarrow \nu(A) = 0$$

holds for each $A \in \mathcal{A}$.

The densities can help with changing the measure in the integral. We have the following result.

Theorem 1.2.4 (Theorem 16.11, [7]). *Let $(\Omega, \mathcal{A})$ be a measruable space and let μ and ν be probability measures defined on $(\Omega, \mathcal{A})$. Assume that ν has a density g with respect to μ and $f : \Omega \to \mathbb{R}$ is an arbitrary $\mathcal{A}$-measruable function. Then the function f is integrable with respect to ν if and only if the function gf is integrable with respect to μ. If f is integrable, then*

$$\int_{\Omega} f d\nu = \int_{\Omega} g f d\mu.$$

We also remind well known Lebesgue's dominated convergence theorem.

Theorem 1.2.5 (Theorem 16.4, [7]). *Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space and $(f_n)_{n\in\mathbb{N}}$ be a sequence of $\mathcal{A}$-measurable functions mappings Ω into $\mathbb{R}$. Assume that the sequence $(f_n)_{n\in\mathbb{N}}$ converges pointwise to a function $f : \Omega \to \mathbb{R}$ μ-almost everywhere. If there exists an integrable function $g : \Omega \to \mathbb{R}$ such that*

$$|f_n(\omega)| \leq g(\omega)$$

for every $\omega \in \Omega$ and $n \in \mathbb{N}$, then f is integrable and

$$\lim_{n\to\infty} \int_{\Omega} f_n d\mu = \int_{\Omega} f d\mu.$$

Until now we presented results from the Lebesgue integral theory – integrated functions had values in $\mathbb{R}$. The Lebesgue construction can be extended on functions reaching values in any separable Banach space. Such a variant of the integral was introduced by S. Bochner in [8]. We will use the Bochner integral in Chapter 4. As we find out below this operator shares many properties with the Lebesgue integral.

In the first step we will define the integral for simple functions.

Definition 1.2.6. Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space and $(X, \|\cdot\|)$ a separable Banach space. We say that an $\mathcal{A}$-measurable function $s : \Omega \to X$ is simple when $s(X)$ is finite.

Simple functions have a standard representation.

Remark 1.2.4. Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space and $(X, \|\cdot\|)$ a separable Banach space. If $s : \Omega \to X$ is a simple function reaching exactly N values, where $N \in \mathbb{N}$, then s can be written as

$$s = \sum_{i=1}^{N} x_i \chi_{A_i},$$

where $x_1, ..., x_N \in X$ are distinct and $A_1, ..., A_N \in \mathcal{A}$ are pairwise disjoint. The above formula is called a *standard representation* of s.

We define the Bochner integral of simple functions in the same way as the Lebesgue integral.

Definition 1.2.7. Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space and $(X, \|\cdot\|)$ a separable Banach space. Assume that $N \in \mathbb{N}$, $x_1, ..., x_N \in X$ are distinct and $A_1, ..., A_N \in \mathcal{A}$ are pairwise disjoint and put $s = \sum_{i=1}^{N} x_i \chi_{A_i}$. We define the Bochner integral of s by the formula

$$\int_\Omega s d\mu = \sum_{i=1}^{N} \mu(A_i) x_i.$$

It is worth adding that each simple function has many different representations. However, one can prove, in the same way as in the case of the Lebesgue integral, that the above value does not depend on the representation of the simple function. Now we are going to define the Bochner integral in general.

Definition 1.2.8. Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space and $(X, \|\cdot\|)$ a separable Banach space. We say that an $\mathcal{A}$-measurable function $f : \Omega \to X$ is *Bochner integrable* (*with respect to* μ) if there exists a sequence $(s_n)_{n \in \mathbb{N}}$ of simple functions mapping Ω into X such that $f = \lim_{n\to\infty} s_n$ μ-almost everywhere and

$$\lim_{n\to\infty} \int_\Omega \|f - s_n\| d\mu = 0.$$

In this case we define the *Bochner integral* of f by

$$\int_\Omega f d\mu := \lim_{n\to\infty} \int_\Omega s_n d\mu.$$

One can check that the last limit appearing in the definition of the Bochner integral is independent on the choice of the sequence of simple functions, i.e. the Bochner integral is well defined.

Remark 1.2.5. Note that the integral

$$\int_\Omega \|f - s_n\| d\mu$$

from the above definition is a Lebesgue integral since values of the norm $\|\cdot\|$ lie in $\mathbb{R}$.

We have the following characterization of Bochner integrable functions by the Lebesgue integral.

Theorem 1.2.6 (Theorem 2 from Chapter 2.2, [15]). *Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space, $(X, \|\cdot\|)$ a separable Banach space and $f : \Omega \to X$ be $\mathcal{A}$-measurable function. Then f is Bochner integrable if and only if $\int_\Omega \|f\| d\mu < \infty$.*

We said that the Lebesgue and Bochner integral have a lot of common properties. In the next theorem we will enumerate these which will be used in Chapter 4.

Theorem 1.2.7 (Theorem 4 from Chapter 2.2, [15]). *Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space and $(X, \|\cdot\|)$ a separable Banach space. If $f : \Omega \to X$ is Bochner integrable, then*

$$\left\| \int_\Omega f d\mu \right\| \leq \int_\Omega \|f\| d\mu.$$

Moreover, for any sequence $(A_n)_{n\in\mathbb{N}}$ of pairwise disjoint sets from $\mathcal{A}$ we have

$$\int_{\bigcup_{n=1}^\infty A_n} f d\mu = \sum_{n=1}^\infty \int_{A_n} f d\mu.$$

The last result of the first chapter allows us to change a measure in the Bochner integral.

Lemma 1.2.1. *Let $(\Omega, \mathcal{A}, \mu)$ be a probability space and $(X, \|\cdot\|)$ a separable Banach space. Assume that ν is a probability measure on the space $(\Omega, \mathcal{A})$ with a density $g : \Omega \to \mathbb{R}$ with respect to μ. If $f : \Omega \to X$ is a bounded Bochner integrable function with respect to μ, then:*
(i) gf is Bochner integrable with respect to μ,
(ii) f is Bochner integrable with respect to ν,
(iii) the equality

$$\int_\Omega g f d\mu = \int_\Omega f d\nu. \tag{1.2}$$

holds.

Proof. First of all we show (i). Take any $M \in (0, \infty)$ such that

$$\|f(\omega)\| < M \quad \text{for all } \omega \in \Omega.$$

Note that gf is $\mathcal{A}$-measurable and

$$\int_\Omega \|gf\| d\mu = \int_\Omega |g| \|f\| d\mu \leq \int_\Omega gM d\mu = M \int_\Omega g d\mu = M < \infty.$$

Using Theorem 1.2.6 we see that gf is Bochner integrable with respect to μ.

We are going to show (ii). Theorem 1.2.4 implies the equality

$$\int_\Omega \|f\| d\nu = \int_\Omega \|gf\| d\mu.$$

Hence and from Theorem 1.2.6 we get integrability of f with respect to ν in the sense of Bochner.

It remains to prove equality (1.2). At the beginning note that this equality is satisfied by simple functions. Let $(s_k)_{k\in\mathbb{N}}$ be a sequence of simple functions from Ω into X such that $\|s_k(\omega)\| < M$ for every $\omega \in \Omega$ and $k \in \mathbb{N}$, $f = \lim_{k\to\infty} s_k$ μ-almost everywhere and

$$\lim_{k\to\infty} \int_\Omega \|f - s_k\| d\mu = 0.$$

Then $gf = \lim_{k\to\infty} gs_k$ μ-almost everywhere and $f = \lim_{k\to\infty} s_k$ ν-almost everywhere since $\nu \ll \mu$. Note that the function gM is Lebesgue integrable with respect to μ and the sequence $\big(\|gf - gs_k\|\big)_{k\in\mathbb{N}}$ is dominated by $2gM$. Using the dominated convergence theorem we get

$$\lim_{k\to\infty} \int_\Omega \|gf - gs_k\| d\mu = 0$$

and, in view of Theorem 1.2.4, we have

$$\lim_{k\to\infty} \int_\Omega \|f - s_k\| d\nu = \lim_{k\to\infty} \int_\Omega \|gf - gs_k\| d\mu = 0.$$

Hence by the definition of the Bochner integral

$$\int_\Omega gf d\mu = \lim_{k\to\infty} \int_\Omega gs_k d\mu \qquad \text{and} \qquad \int_\Omega f d\nu = \lim_{k\to\infty} \int_\Omega s_k d\nu.$$

Therefore

$$\int_\Omega gf d\mu = \lim_{k\to\infty} \int_\Omega gs_k d\mu = \lim_{k\to\infty} \int_\Omega s_k d\nu = \int_\Omega f d\nu. \qquad \square$$

The above lemma seems to be as trivial that it should be known before but I did not meet this result in the literature and its proof has been included for the sake of completeness.

Chapter 2

The archetypal equation

In this part we are working with the archetypal equation (2.1). Chapter 2 is splited into four sections. In the first we introduce the archetypal equation and consider its particular forms. In Section 2.2 we formulate a problem posed by Gregory Derfel and explain his motivation to propose equation (2.8). Section 2.3 contains a partial solution of this problem; in particular, we prove that every bounded continuous solution of Derfel's equation (2.8), attaining the global extremum, is constant. Section 2.4 is devoted to solutions of this equation which do not attain their global extremum.

Section 2.1 and Section 2.2 are based on the papers [10] and [11]. Section 2.3 and 2.4 contain results from [37] which is an extension of the article [36].

2.1 The archetypal equation and its particular forms

The first functional equation described in the thesis is the *archetypal equation*. It was introduced and examined by L. Bogachev, G. Derfel and S. Molchanov in [10] and [11] from 2015. In fact, the analysis of the archetypal equation began much earlier. We would like to pay attention on the paper [13] written by G. Derfel in 1989. There he found a connection between bounded continuous solutions of the archetypal equation and the Grintsevichyus series (see [18]).

In this chapter we treat a probability measure $\mu : \mathcal{B}(\mathbb{R}^2) \to [0,1]$ as fixed. Remind that the archetypal equation is a functional equation of the form

$$\varphi(x) = \iint_{\mathbb{R}^2} \varphi\big(a(x-b)\big)\mu(da, db). \tag{2.1}$$

Furthermore, we fix an arbitrary probability space $(\Omega, \mathcal{A}, P)$ and a random vector $(\alpha, \beta) : \Omega \to \mathbb{R}^2$. We assume that the distribution of (α, β) is equal to μ. Then equation (2.1) can be written in the language of random variables as

$$\varphi(x) = \int_{\Omega} \varphi\big(\alpha(\omega)(x - \beta(\omega))\big) P(d\omega). \tag{2.2}$$

Since the distribution of (α, β) equals to μ, equations (2.1) and (2.2) are equivalent. In the present chapter we use both forms alternatively.

Remark 2.1.1. At the beginning note that every constant function is a solution of the archetypal equation since the measure μ is a probabilistic one. We are interested in finding an answer to the question for which measures there are only constant solutions of equation (2.1) in the class of bounded continuous functions. The restriction to this class is necessary if we want to exclude the following pathological cases.

Example 2.1.1. Suppose that $\mu(\{(1,1)\}) = \mu(\{(2,1)\}) = \frac{1}{2}$. Then equation (2.1) reduces to

$$\varphi(x) = \frac{1}{2}\varphi(x-1) + \frac{1}{2}\varphi(2x-2).$$

We will see later that it is the so-called *degenerated case* of the archetypal equation and it has no non-constant solutions in the class of bounded continuous functions. However, the function $\varphi : \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}, \end{cases}$$

is its discontinuous solution.

L. Bogachev, G. Derfel and S. Molchanov called equation (2.1) as archetypal since it is a rich source of many famous and studied earlier functional equations with rescaling. The measure μ determines the shape of equation (2.1). For instance if we take any discrete measure with a finite support, then we get an equation of finite order. If we assume that the support of the measure μ is contained in the line $\{1\} \times \mathbb{R}$, then we will get the integrated Cauchy functional equation described below.

Example 2.1.2 (Bogachev, Derfel, Molchanov, [11])**.** If we consider a measure μ which support is contained in the line $\{1\} \times \mathbb{R}$, then equation (2.1) reduces to

$$\varphi(x) = \int_{\mathbb{R}} \varphi(x - t)\nu(dt),$$

where $\nu : \mathcal{B}(\mathbb{R}) \to [0,1]$ is a Borel probability measure. In the literature the above equation is called as the *integrated Cauchy functional equation* and plays an important role in many branches of mathematics. Its solutions have been described in the famous Choquet-Deny Theorem – the reader is refered to the original paper [12] from 1960 (see also [30] and [31]).

There is a more surprising fact that the archetypal equation contains, as particular cases, some differential equations. In the next proposition we will present (reasoning comes from [11]) how to obtain from (2.2) the pantograph equation (see [28] for its definition).

Proposition 2.1.1 (Bogachev, Derfel, Molchanov, [11]). *Let N be a natural number, $p_1, ..., p_N \in (0,1)$ and $a_1, ..., a_N \in \mathbb{R} \setminus \{0\}$. Assume that $\sum_{i=1}^{N} p_i = 1$ and $\varphi : \mathbb{R} \to \mathbb{R}$ is a bounded continuous function. Then φ is a solution of the equation*

$$\varphi(x) = \sum_{i=1}^{N} p_i \int_0^{\infty} \varphi\big(a_i(x-t)\big)e^{-t}dt \tag{2.3}$$

if and only if φ is differentiable and satisfies the pantograph equation

$$\varphi'(x) + \varphi(x) = \sum_{i=1}^{N} p_i \varphi(a_i x). \tag{2.4}$$

Proof. First of all assume that φ satisfies equation (2.3). Using the substitution $u = x - t$ we get

$$\int_0^{\infty} \varphi\big(a_i(x-t)\big)e^{-t}dt = -\int_x^{-\infty} \varphi(a_i u)e^{u-x}du = e^{-x}\int_{-\infty}^{x} \varphi(a_i u)e^{u}du$$

for all $i \in \{1, 2, ..., N\}$. The function φ is bounded and continuous, and thus the right-hand side of the equality

$$\varphi(x) = \sum_{i=1}^{N} p_i e^{-x} \int_{-\infty}^{x} \varphi(a_i u)e^{u}du$$

is differentiable and

$$\varphi'(x) = \sum_{i=1}^{N} p_i \left(\varphi(a_i x) - e^{-x}\int_{-\infty}^{x} \varphi(a_i u)e^{u}du\right)$$

for every $x \in \mathbb{R}$. Summing up two above equalities we obtain

$$\varphi'(x) + \varphi(x) = \sum_{i=1}^{N} p_i \varphi(a_i x)$$

for all $x \in \mathbb{R}$, that is φ satisfies the pantograph equation.

Now assume that φ is differentiable and satisfies (2.4). If we multiply both sides of equality (2.4) by e^u, then we get

$$\varphi'(u)e^u + \varphi(u)e^u = \sum_{i=1}^{N} p_i \varphi(a_i u)e^u. \tag{2.5}$$

Integrating both sides of this equality from 0 to x, we can write the left-hand side of the obtained equality as

$$\int_0^x \big(\varphi'(u)e^u + \varphi(u)e^u\big)du = \int_0^x \big(\varphi(u)e^u\big)'du = \varphi(x)e^x - \varphi(0).$$

Therefore, after integration, equality (2.5) can be written as

$$\varphi(x)e^x - \varphi(0) = \sum_{i=1}^{N} p_i \int_0^x \varphi(a_i u)e^u du.$$

Since the function φ is bounded, we have $\varphi(x)e^x \to 0$ if $x \to -\infty$. This fact, jointly with the above equality, implies that $\varphi(0) = \sum_{i=1}^{N} p_i \int_{-\infty}^{0} \varphi(a_i u)e^u du$. Hence

$$\begin{aligned}
\varphi(x) =& \varphi(0)e^{-x} + \sum_{i=1}^{N} p_i e^{-x} \int_0^x \varphi(a_i u)e^u du \\
=& \sum_{i=1}^{N} p_i \int_{-\infty}^{0} \varphi(a_i u)e^{u-x} du + \sum_{i=1}^{N} p_i \int_0^x \varphi(a_i u)e^{u-x} du \\
=& \sum_{i=1}^{N} p_i \int_{-\infty}^{x} \varphi(a_i u)e^{u-x} du
\end{aligned}$$

for all $x \in \mathbb{R}$. Using the substitution $u = x - t$ we come to

$$\varphi(x) = \sum_{i=1}^{N} p_i \int_0^{\infty} \varphi\big(a_i(x-t)\big)e^{-t}dt, \qquad x \in \mathbb{R},$$

and the proof is complete. □

How to obtain other differential equations, even those of higher order, the interested reader can get to know in [11].

2.2 A dichotomy and a problem of Gregory Derfel

2.2.1 Degenerated forms of the archetypal equation

In this part of the thesis we will see three degenerated forms of equation (2.1). The results included in this subsection are presented without their proofs – the reader can find the details in [11]. These cases will be excluded from further considerations.

We say that the first degenerated case of equation (2.2) occurs if the random variable α takes the value 0 with positive probability.

Theorem 2.2.1 (Bogachev, Derfel, Molchanov, [11])**.** *Suppose that* $P(\alpha = 0) > 0$. *Then any bounded solution* $\varphi : \mathbb{R} \to \mathbb{R}$ *of equation* (2.2) *is constant.*

The second degenerated case can be treated as an extension of the Choquet-Deny Theorem. Before we formulate this result we remind the definition of the arithmetic distribution.

Definition 2.2.1. We say that a random variable $X : \Omega \to \mathbb{R}$ has the *arithmetic distribution* if there exists $\lambda \in [0, \infty)$ such that the support of the distribution of X is contained in $\lambda\mathbb{Z}$. The largest λ with such a property is called a *span*.

Now we are ready to present a generalization of the Choquet-Deny theorem.

Theorem 2.2.2 (Bogachev, Derfel, Molchanov, [11])**.** *Suppose that* $P(|\alpha| = 1) = 1$ *and* $P(\alpha = -1) > 0$. *Let* β^+ *and* β^- *denote random variables which distributions are equal to the conditional distribution of* β *given by* $\alpha = 1$ *and* $\alpha = -1$, *respectively* (*in the case* $\alpha = -1$, *we set* $\beta^+ = 0$).

(*a*) *If a distribution of* β^+ *is non-arithmetic, then every bounded continuous solution* $\varphi : \mathbb{R} \to \mathbb{R}$ *of equation* (2.2) *is constant.*

(*b*) *Let the distribution of* β^+ *be arithmetic with span* $\lambda \in (0, \infty)$.

$(b - i)$ *If the distribution of* β^- *is not supported on any set* $\lambda_0 + \lambda\mathbb{Z}$, *where* $\lambda_0 \in \mathbb{R}$, *then every bounded continuous solution* $\varphi : \mathbb{R} \to \mathbb{R}$ *of equation* (2.2) *is constant.*

$(b - ii)$ *Otherwise, the general bounded continuous solution* $\varphi : \mathbb{R} \to \mathbb{R}$

of equation (2.2) *is of the form* $\varphi(x) = g(x/\lambda)$, *where* $g : \mathbb{R} \to \mathbb{R}$ *is a continuous, 1-periodic function symmetric about the point* $x_0 = \lambda_0/2\lambda$.

The above theorem is an extension of the Choquet-Deny Theorem on the case when we accept both values 1 and -1 for the rescaling parameter. If the assumption saying that $P(\alpha = -1) > 0$ is not satisfied, then the above theorem reduces to the classical Choquet-Deny Theorem.

It is easy to check that in the case like this every function g from the part $(b - ii)$ is a solution of the archetypal equation. However, the most important statement of the above theorem is that there are no other bounded continuous solutions – in this case we have a full description of non-constant bounded continuous solutions.

We are going to present the last degenerated case of the archetypal equation. We accept the following definition.

Definition 2.2.2. The random variables α and β are said to be *in resonance* if there exists a constant $c \in \mathbb{R}$ such that $P\big(\alpha(c - \beta) = c\big) = 1$.

Meaning of this definition is as follows: almost all affine functions from equation (2.2) have the same fixed point. If we choose random variables in such a way, then we get

Theorem 2.2.3 (Bogachev, Derfel, Molchanov, [11]). *Let* $P(|\alpha| \neq 1) > 0$ *and suppose that* α *and* β *are in resonance. Then any bounded continuous solution* $\varphi : \mathbb{R} \to \mathbb{R}$ *of equation* (2.2) *is constant.*

Remark 2.2.1. As we said at the beginning of this section, the degenerated cases will be excluded from the further analysis. Therefore we will say that the hypothesis (H) is satisfied if

$$P(\alpha = 0) = 0, \qquad P(|\alpha| = 1) < 1 \quad \text{and} \quad \alpha \text{ and } \beta \text{ are not in resonance.}$$

2.2.2 Parameter K and its influence on the set of bounded continuous solutions of the archetypal equation

As was observed by Gregory Derfel in his paper [13] from 1989, a behaviour of bounded continuous solutions of the archetypal equation depends very strongly on the value of the parameter K, which is defined as

$$K := \iint_{\mathbb{R}^2} \ln|a| \mu(da, db) = \int_{\Omega} \ln|\alpha(\omega)| P(d\omega). \tag{2.6}$$

The next crucial theorem shows the importance of this number.

Theorem 2.2.4 (Bogachev, Derfel, Molchanov, [10]). *Assume hypothesis* (H) *and suppose that the integrals* K *and* $\iint_{\mathbb{R}^2} \ln(\max(|b|, 1))\mu(da, db)$ *are finite.*

(i) If $K < 0$*, then any bounded continuous solution of the archetypal equation is constant.*

(ii) If $K > 0$ *and, in addition* $\alpha > 0$ *a.s., then there exists a non-constant bounded continuous solution of the archetypal equation.*

The solution mentioned in Theorem 2.2.4 (ii) is called the *canonical solution* and it is connected with the notion of Grintsevichyus series (see [18]). We denote this special solution by F_Υ. It is easy to note that any linear combination of the canonical and constant solution still satisfies the archetypal equation. Under some technical assumptions any bounded continuous solution of (2.1) has to be such a linear combination in the case $\alpha > 0$ a.s. – see Theorem 4.3 from [10]. It is worth mentioining that solutions of this form may do not cover the set of all bounded continuous solutions of the archetypal equation in general. In [25] the authors obtained non-constant solutions of a pantograph equation different from $aF_\Upsilon + b$, where $a, b \in \mathbb{R}$.

It is worth adding that F_Υ is no longer a solution of the archetypal equation in the case $P(\alpha < 0) > 0$. For more details the reader is refered to [10] and [11].

The case when

$$P(\alpha < 0) > 0 \tag{2.7}$$

is not well explored yet. We have a small knowledge about bounded continuous solutions of the archetypal equation is this case. There is only one exception when we can give a full description of bounded continuous solutions, namely if $P(|\alpha| = 1) = 1$, what was made in Theorem 2.2.2.

Note that if $|\alpha| = 1$, then $K = 0$. If we assume the hypothesis (H), inequality (2.7) and $K > 0$, then we know practically nothing about the existence of non-constant bounded continuous solutions of (2.1). It was a motivation for Gregory Derfel to pose the following problem during the *21st European Conference on Iteration Theory* which held in 2016 in Innsbruck (Austria).

Problem Is there any non-constant bounded continuous solution $\varphi : \mathbb{R} \to \mathbb{R}$ of the functional equation

$$\varphi(x) = \frac{1}{2}\varphi(x-1) + \frac{1}{2}\varphi(-2x)? \tag{2.8}$$

This question was repeated by him during the *55th International Symposium on Functional Equations* in Chengdu (China) next year. If we look more carefuly on equation (2.8), then we will see that in this case we have

$$K = \frac{1}{2}\ln|1| + \frac{1}{2}\ln|-2| = \frac{\ln 2}{2} > 0.$$

Equation (2.8) examplifies the situation when the rescaling parameter attains negative values and K is positive.

The goal of my scientific work is to find bounded continuous solutions of this equation and enhance the knowledge for this class of functional equations.

2.3 Solutions attaining the global extremum

In this section, in the case $\mu\big((-\infty,0)\times\mathbb{R}\big) > 0$, we examine solutions of equation (2.1) attaining the global extremum. It turns out that such solutions must be constant for a wide class of probability measures. I remind that all results appearing in this section come from my article [37].

Let us begin with a simple observation.

Remark 2.3.1. We have seen that each constant function is a solution of the archetypal equation. Furthermore, note that the linear combination of solutions of (2.1) is still its solution. Hence, studying solutions $\varphi:\mathbb{R}\to\mathbb{R}$ of (2.1) which attain their global extremum we can assume without loss of generality that φ is non-negative and $\min\varphi(\mathbb{R}) = 0$. If, in addition, φ is bounded, then we can confine reasonings to the case $\sup\varphi(\mathbb{R}) \leq 1$ without loss of generality.

We start with the following technical lemma which will be our basic tool in this section.

Lemma 2.3.1 (Lemma 2.2, [37]). *Let $\varphi:\mathbb{R}\to[0,\infty)$ be a continuous solution of equation* (2.1) *and let $x_0\in\mathbb{R}$ be such that $\varphi(x_0)=0$. Then*

$$\varphi\left(c_p c_{p-1}\cdot\ldots\cdot c_1 x_0 - \sum_{i=1}^{p} c_p\cdot\ldots\cdot c_i d_i\right) = 0$$

for all $p\in\mathbb{N}$ and $(c_1,d_1),\ldots,(c_p,d_p)\in\operatorname{supp}\mu$.

Proof. Let $p \in \mathbb{N}$ be fixed and assume that $(c_1, d_1), ..., (c_p, d_p) \in \operatorname{supp}\mu$. Applying equality (2.1) to x_0 we get

$$0 = \varphi(x_0) = \iint_{\mathbb{R}^2} \varphi\big(a(x_0 - b)\big)\mu(da, db).$$

Since φ is continuous and non-negative, we have

$$\varphi\Big(a(x_0 - b)\Big) = 0 \quad \text{ for all } (a, b) \in \operatorname{supp}\mu.$$

In particular, we get $\varphi\Big(c_1x_0 - c_1d_1\Big) = 0$. If we repeat this reasoning $p-1$ times to the points $c_j \cdot ... \cdot c_1x_0 - \sum_{i=1}^{j} c_j \cdot ... \cdot c_id_i$ and the pairs (c_{j+1}, d_{j+1}), $j = 1, ..., p-1$, in turn, we obtain the assertion. □

We will prove also a lemma which asserts that some subsets are dense in $\mathbb{R}$. The statement is obvious but it can help to pay attention on the crucial properties of sets appearing in the proof of Theorem 2.3.1. Further a simple proof of this lemma is presented for the sake of completeness.

Lemma 2.3.2 (Lemma 2.3, [37]). *Let $t \in \mathbb{R} \setminus \{0\}$, $v \in (1, \infty)$ and $(u_n)_{n\in\mathbb{N}}$ be a sequence of reals. Then the set $D = \bigcup_{n=1}^{\infty} D_n$, where*

$$D_n := \left\{u_n + \frac{st}{v^n} : \quad s \in \mathbb{Z}\right\} \quad \text{ for all } n \in \mathbb{N},$$

is dense in $\mathbb{R}$.

Proof. Without loss of generality we may consider only the case when t is positive. Fix $a, b \in \mathbb{R}$ such that $a < b$ and define the interval $I = (a, b)$. We have to show that $I \cap D$ is nonempty. Take $n_0 \in \mathbb{N}$ such that

$$\frac{t}{v^{n_0}} < b - a. \tag{2.9}$$

Let s_0 be the greatest member of $\mathbb{Z}$ for which

$$u_{n_0} + \frac{s_0t}{v^{n_0}} \leq a.$$

Hence, by the definition of the number s_0, we have

$$u_{n_0} + \frac{s_0t + t}{v^{n_0}} > a.$$

Moreover, using a definition of s_0 and inequality (2.9) we get

$$u_{n_0} + \frac{s_0 t + t}{v^{n_0}} = u_{n_0} + \frac{s_0 t}{v^{n_0}} + \frac{t}{v^{n_0}} \leq a + \frac{t}{v^{n_0}} < b.$$

Therefore $u_{n_0} + \frac{s_0 t+t}{v^{n_0}} \in I$. This means that D is dense in $\mathbb{R}$. $\square$

We are in a position to formulate and prove the result which provides a partial answer to the Derfel's question. This is the first theorem of two main results of the paper [37].

Theorem 2.3.1 (Theorem 2.4, [37]). *Assume that* $\mu\big(((-\infty,0)\setminus\{-1\})\times\mathbb{R}\big) > 0$ *and there exists* $b \in \mathbb{R}\setminus\{0\}$ *such that* $(1,b) \in \operatorname{supp}\mu$.
(i) If $\operatorname{supp}\mu \cap \big((\mathbb{R}\setminus\mathbb{Z})\times\mathbb{R}\big) \neq \emptyset$, *then every continuous solution of equation* (2.1) *attaining its global extremum is constant.*
(ii) If $\operatorname{supp}\mu \subseteq \mathbb{Z}\times\mathbb{R}$, *then every bounded continuous solution of equation* (2.1) *attaining its global extremum is constant.*

Proof. The pair $(1,b)$ will be denoted by (a_0,b_0) from this point. Take any continuous solution $\varphi : \mathbb{R} \to \mathbb{R}$ of (2.1) attaining its global extremum. In view of Remark 2.3.1 we can assume that φ is non-negative and there exists $x_0 \in \mathbb{R}$ such that $\varphi(x_0) = 0$. Since φ is continuous, it is sufficient to show that φ takes value 0 on some dense subset D of the real line. The proof is splitted into three general cases and we will describe how to obtain the desired dense set in each of them. We shall examine each of the following situations:

I $\operatorname{supp}\mu \cap \big((\mathbb{R}\setminus\mathbb{Q})\times\mathbb{R}\big) \neq \emptyset$,

II $\operatorname{supp}\mu \cap \big((\mathbb{Q}\setminus\mathbb{Z})\times\mathbb{R}\big) \neq \emptyset$,

III $\operatorname{supp}\mu \subseteq \mathbb{Z}\times\mathbb{R}$.

Furthermore we distinguish two complementary subcases for both case I and case II.

I.A There exist $a \in (\mathbb{R}\setminus\mathbb{Q}) \cap (-\infty,0)$ and $b \in \mathbb{R}$ such that $(a,b) \in \operatorname{supp}\mu$.

I.B If $(a,b) \in \operatorname{supp}\mu$ and $a \in \mathbb{R}\setminus\mathbb{Q}$, then a is positive.

II.A There exist $a \in (\mathbb{Q}\setminus\mathbb{Z}) \cap (-\infty,0)$ and $b \in \mathbb{R}$ such that $(a,b) \in \operatorname{supp}\mu$.

II.B If $(a,b) \in \operatorname{supp}\mu$ and $a \in \mathbb{Q}\setminus\mathbb{Z}$, then a is positive.

In summary, we obtain the five cases (I.A, I.B, II.A, II.B and III) and note that the measure μ fulfills at least one of them. We shall prove that in each one the function φ is constant.

I.A. Assume that there exists $(a, b) \in \mathrm{supp}\mu$ such that $a \in (-\infty, 0) \setminus \mathbb{Q}$. Put $(a_1, b_1) := (a, b)$ and remember that $a_0 = 1$ and $b_0 \neq 0$. Fix also arbitrary $k, l \in \mathbb{N}$. If we apply Lemma 2.3.1 with $(c_i, d_i) = (a_0, b_0)$, where $i = 1, 2, ..., k$, then we obtain the equality

$$\varphi(x_0 - kb_0) = 0$$

since $a_0 = 1$. Applying equality (2.1) for $x_0 - kb_0$ one can deduce that

$$\varphi\big(a_1(x_0 - b_1) - a_1 kb_0\big) = 0.$$

Finally, if we use Lemma 2.3.1 with $(c_i, d_i) = (a_0, b_0)$, where $i = 1, 2, ..., l$, to the point $a_1(x_0 - b_1) - a_1 kb_0$, then we will get

$$\varphi\Bigg(a_1(x_0 - b_1) - (ka_1 + l)b_0\Bigg) = 0.$$

The Kronecker density theorem [19, Chapter XXIII] asserts that the set

$$\big\{ka_1 + l : \quad k, l \in \mathbb{N}\big\}$$

is dense in $\mathbb{R}$. Hence

$$\big\{a_1(x_0 - b_1) - (ka_1 + l)b_0 : \quad k, l \in \mathbb{N}\big\}$$

is also dense as $b_0 \neq 0$. This implies that φ is constant and the proof is complete in this subcase.

I.B. This case means that if $(a, b) \in \mathrm{supp}\mu$ and a is negative, then $a \in \mathbb{Q}$. Since $\mu\big((-\infty, 0) \times \mathbb{R}\big) > 0$ there exists $(a, b) \in \mathrm{supp}\mu$ with $a < 0$. Put $(a_1, b_1) := (a, b)$. Then $a_1 \in \mathbb{Q}$. Take also $(a_2, b_2) \in \mathrm{supp}\mu$ with $a_2 \in \mathbb{R} \setminus \mathbb{Q}$. Then $a_2 > 0$. As before we obtain the equality $\varphi\big(a_1(x_0 - b_1) - a_1 kb_0\big) = 0$, where $k \in \mathbb{N}$ is fixed. In the next step, applying equality (2.1) to $a_1(x_0 - b_1) - a_1 kb_0$, we get

$$\varphi\big(a_1 a_2(x_0 - b_1) - a_2 b_2 - a_1 a_2 kb_0\big) = 0.$$

Fix an arbitrary $l \in \mathbb{N}$. Finally, we apply Lemma 2.3.1 with $(c_i, d_i) = (a_0, b_0)$ for $i = 1, ..., l$ to the point $a_1 a_2(x_0 - b_1) - a_2 b_2 - a_1 a_2 kb_0$. As a result we have

$$\varphi\big(a_1 a_2(x_0 - b_1) - a_2 b_2 - (l + a_1 a_2 k)b_0\big) = 0.$$

Note that $a_1 a_2$ is a negative irrational number, so again using the Kronecker density theorem we have constructed a dense subset D of $\mathbb{R}$ while k, l runs through $\mathbb{N}$. The proof is complete in this case.

II. Now we assume that $\mathrm{supp}\mu \cap \big((\mathbb{Q} \setminus \mathbb{Z}) \times \mathbb{R}\big) \neq \emptyset$. Since we are done in case I we may assume that $\mathrm{supp}\mu \subseteq \mathbb{Q} \times \mathbb{R}$. Let $A \subseteq \mathbb{Q}$ be the smallest set such that $\mathrm{supp}\mu \subseteq A \times \mathbb{R}$, i.e.

$$A = \{a \in \mathbb{Q} : \quad (a, b) \in \mathrm{supp}\mu \text{ for some real } b\}.$$

Then there is a set $I \subseteq \mathbb{N} \cup \{0\}$ and an injective sequence $(a_i)_{i \in I}$ such that $A = \{a_i : i \in I\}$. Moreover, we can take $I = \{0, 1, ..., n-1\}$ if A has exactly n elements and $I = \mathbb{N} \cup \{0\}$ when A is infinite. Observe that the condition $\mathrm{supp}\mu \cap ((\mathbb{Q} \setminus \mathbb{Z}) \times \mathbb{R}) \neq \emptyset$, assumed in the present case II, means that $A \setminus \mathbb{Z} \neq \emptyset$. For every $i \in I$ we define also a Borel measure μ_i by the equality

$$\mu_i(B) = \mu(\{a_i\} \times B) \quad \text{for each Borel } B \subseteq \mathbb{R}.$$

Note that for any $i \in I$ we have $\mu_i(\mathbb{R}) \in (0, 1)$ and $\sum_{i \in I} \mu_i(\mathbb{R}) = 1$. Moreover, equation (2.1) can be rewritten in the form

$$\varphi(x) = \sum_{i \in I} \int_{\mathbb{R}} \varphi\big(a_i(x - b)\big)\mu_i(db). \tag{2.10}$$

II.A. In this case there exists $i \in I$ such that a_i is negative and non-integer. First of all remember that $a_0 = 1$ and $b_0 \neq 0$. Without loss of generality we may assume that $i = 1$. In other words $a_1 \in (-\infty, 0) \cap (\mathbb{Q} \setminus \mathbb{Z})$. Then there exist coprime $q, q_0 \in \mathbb{Z}$ such that $a_1 = q/q_0$ and $q < 0$. The definitions of q, q_0 and a_1 imply that $q_0 \geq 2$. We fix also any $b_1 \in \mathbb{R}$ for which $(a_1, b_1) \in \mathrm{supp}\mu$. Define the sequence $(D_n)_{n \in \mathbb{N}}$ of sets putting

$$D_n := \left\{ a_1^n x_0 - \sum_{i=1}^{n} a_1^i b_1 - \frac{s b_0}{q_0^n} : \quad s \in \mathbb{Z} \right\}$$

and let

$$D := \bigcup_{n=1}^{\infty} D_n.$$

Since $b_0 \neq 0$ and $q_0 > 1$, the density of the set D is just a consequence of Lemma 2.3.2.

Our goal is to prove that the function φ vanishes on D. We shall use the mathematical induction with respect to n. In the first step we show that $\varphi_{|D_1} = 0$, i.e.

$$\varphi\left(a_1(x_0 - b_1) - \frac{sb_0}{q_0}\right) = 0 \quad \text{for all } s \in \mathbb{Z}.$$

Fix any $k, l \in \mathbb{N} \cup \{0\}$. At the beginning note that we can obtain the equality

$$0 = \varphi\left(a_1(x_0 - b_1) - a_1 k b_0\right) = \varphi\left(a_1(x_0 - b_1) - \frac{qk}{q_0} b_0\right)$$

in the same way as in point I.A. If we apply Lemma 2.3.1 with $(c_j, d_j) = (a_0, b_0)$, where $j = 1, 2, ..., l$, to the point $a_1(x_0 - b_1) - \frac{qk}{q_0} b_0$, then we will get

$$\varphi\left(a_1(x_0 - b_1) - l b_0 - \frac{qk}{q_0} b_0\right) = \varphi\left(a_1(x_0 - b_1) - \frac{q_0 l + qk}{q_0} b_0\right) = 0. \tag{2.11}$$

Note that the expression $q_0 l + qk$ runs through the whole $\mathbb{Z}$, while $k, l \in \mathbb{N} \cup \{0\}$, since q and q_0 are coprime and of different signs. Therefore equality (2.11) implies that

$$\varphi(x) = 0 \quad \text{for every } x \in D_1.$$

Let $n \in \mathbb{N}$ be fixed and assume that $\varphi_{|D_n} = 0$. We prove that $\varphi_{|D_{n+1}} = 0$. Take any $s \in \mathbb{Z}$, put

$$r = a_1^{n+1} x_0 - \sum_{i=1}^{n+1} a_1^i b_1 - \frac{s b_0}{q_0^{n+1}}$$

and observe that $r \in D_{n+1}$. Since q and q_0 are coprime and of different signs, we can find $k, l \in \mathbb{N} \cup \{0\}$ such that $q_0^{n+1} l + qk = s$. If we use the induction hypothesis and equality (2.10), then we will get

$$0 = \varphi\left(a_1^n x_0 - \sum_{i=1}^{n} a_1^i b_1 - \frac{k b_0}{q_0^n}\right) = \sum_{j \in I} \int_{\mathbb{R}} \varphi\left(a_j\left(a_1^n x_0 - \sum_{i=1}^{n} a_1^i b_1 - \frac{k b_0}{q_0^n} - b\right)\right) \mu_j(db),$$

and thus, as φ is non-negative,

$$\int_{\mathbb{R}} \varphi\left(a_j\left(a_1^n x_0 - \sum_{i=1}^{n} a_1^i b_1 - \frac{k b_0}{q_0^n} - b\right)\right) \mu_j(db) = 0 \quad \text{for all } j \in I.$$

In particular, for $j = 1$ we have

$$\int_{\mathbb{R}} \varphi\left(a_1\left(a_1^n x_0 - \sum_{i=1}^{n} a_1^i b_1 - \frac{kb_0}{q_0^n} - b\right)\right)\mu_1(db) = 0.$$

The function φ is non-negative and continuous. Hence

$$\varphi\left(a_1\left(a_1^n x_0 - \sum_{i=1}^{n} a_1^i b_1 - \frac{kb_0}{q_0^n} - b\right)\right) = 0 \quad \text{for every } b \in \operatorname{supp}\mu_1.$$

Taking $b = b_1$ in the above equality we get

$$\varphi\left(a_1^{n+1} x_0 - \sum_{i=1}^{n} a_1^{i+1} b_1 - a_1 b_1 - a_1\frac{kb_0}{q_0^n}\right) = 0.$$

Thus, using the representation $a_1 = q/q_0$, we come to the equality

$$\varphi\left(a_1^{n+1} x_0 - \sum_{i=1}^{n+1} a_1^i b_1 - \frac{qkb_0}{q_0^{n+1}}\right) = 0.$$

Remember that $a_0 = 1$ and the numbers k and l were chosen such that $q_0 l^{n+1} + qk = s$. If we apply Lemma 2.3.1 with $(c_j, d_j) = (a_0, b_0)$, where $j = 1, 2, ..., l$, to the point from the last equality, then we will get

$$\varphi\left(a_1^{n+1} x_0 - \sum_{i=1}^{n+1} a_1^i b_1 - \frac{qkb_0}{q_0^{n+1}} - lb_0\right) = 0,$$

that is

$$\varphi\left(a_1^{n+1} x_0 - \sum_{i=1}^{n+1} a_1^i b_1 - \frac{qk + q_0^{n+1} l}{q_0^{n+1}} b_0\right) = 0.$$

The above equality means that $\varphi(r) = 0$. Since r was an arbitrary element of D_{n+1}, we have proven that

$$\varphi(x) = 0 \quad \text{for every } x \in D_{n+1}.$$

Consequently, by the mathematical induction, we get $\varphi_{|D} = 0$ and the proof has been completed in case II.A.

II.B. In this case $A \setminus \mathbb{Z} \neq \emptyset$ and each negative element of A is an integer. Remember that we have $a_0 = 1$ and $b_0 \neq 0$. We know that $\mu\big((-\infty, 0) \times \mathbb{R}\big) > 0$.

This condition implies that we can choose $i \in I$ such that $a_i \in (-\infty, 0)$. All such numbers are, in view of the assumptions of case II.B, integers. Without loss of generality we may assume that $i = 1$. Then $a_1 \in \mathbb{Z}$. Take an arbitrary $b_1 \in \mathrm{supp}\mu_1$. Since $\emptyset \neq A \setminus \mathbb{Z} \subseteq \mathbb{Q} \setminus \mathbb{Z}$ we may assume that $a_2 \in \mathbb{Q} \setminus \mathbb{Z}$. Take any $b_2 \in \mathrm{supp}\mu_2$. Then a_2 must be positive. We define also a parameter $m \in \mathbb{N}$ such that $a_1 a_2^m \in \mathbb{Q} \setminus \mathbb{Z}$ since it may happen that $a_1 a_2 \in \mathbb{Z}$. There exist coprime $q, q_0 \in \mathbb{Z}$ for which $a_1 a_2^m = q/q_0$ and $q < 0$. Then $q_0 \geq 2$. The proof in this case is analogous to that of the previous subcase. This time a sequence of sets $(D_n)_{n \in \mathbb{N}}$ will be defined by

$$D_n := \left\{ (a_1 a_2^m)^n x_0 - \sum_{i=1}^{n} (a_1 a_2^m)^i b_1 - \sum_{i=0}^{n-1} \sum_{j=1}^{m} a_1^i a_2^{im+j} b_2 - \frac{s b_0}{q_0^n} : \quad s \in \mathbb{Z} \right\}$$

and again we put

$$D = \bigcup_{n=1}^{\infty} D_n.$$

We have $b_0 \neq 0$ and $q_0 > 1$. Define a sequence $(u_n)_{n \in \mathbb{N}}$ by

$$u_n = (a_1 a_2^m)^n x_0 - \sum_{i=1}^{n} (a_1 a_2^m)^i b_1 - \sum_{i=0}^{n-1} \sum_{j=1}^{m} a_1^i a_2^{im+j} b_2.$$

Then, in view of Lemma 2.3.2, we will get the density of D. Again we shall use the mathematical induction to prove that φ is constant on D. First of all we are going to show that $\varphi_{|D_1} = 0$. As previously we start with the equality

$$\varphi\big(a_1(x_0 - b_1) - a_1 k b_0\big) = 0,$$

where $k \in \mathbb{N} \cup \{0\}$ is fixed. Applying Lemma 2.3.1 with $(c_j, d_j) = (a_2, b_2)$, where $j = 1, 2, ..., m$, to the point $a_1(x_0 - b_1) - a_1 k b_0$ we get

$$\varphi\left(a_1 a_2^m (x_0 - b_1) - \sum_{i=1}^{m} a_2^i b_2 - a_1 a_2^m k b_0 \right) = 0.$$

Since $a_1 a_2^m = q/q_0$, we have

$$\varphi\left(a_1 a_2^m (x_0 - b_1) - \sum_{i=1}^{m} a_2^i b_2 - \frac{qk}{q_0} b_0 \right) = 0.$$

Fix any $l \in \mathbb{N} \cup \{0\}$. If we use Lemma 2.3.1 with $(c_j, d_j) = (a_0, b_0)$, where $j = 1, ..., l$, to the point from the above equality, then we will obtain

$$\varphi\left(a_1 a_2^m (x_0 - b_1) - \sum_{i=1}^{m} a_2^i b_2 - \frac{qk}{q_0} b_0 - l b_0 \right) = 0,$$

that is

$$\varphi\left(a_1 a_2^m (x_0 - b_1) - \sum_{i=1}^{m} a_2^i b_2 - \frac{qk + q_0 l}{q_0} b_0 \right) = 0.$$

Note that q and q_0 are coprime and they have different signs which implies that the expression $qk + q_0 l$ can take any integer value when k, l run through $\mathbb{N} \cup \{0\}$. Hence

$$\varphi(x) = 0 \quad \text{for every } x \in D_1.$$

Fix $n \in \mathbb{N}$ and assume that $\varphi_{|D_n} = 0$. We will show the equality $\varphi_{|D_{n+1}} = 0$. Take any $s \in \mathbb{Z}$ and put

$$r := (a_1 a_2^m)^{n+1} x_0 - \sum_{i=1}^{n+1} (a_1 a_2^m)^i b_1 - \sum_{i=0}^{n} \sum_{j=1}^{m} a_1^i a_2^{im+j} b_2 - \frac{s b_0}{q_0^{n+1}}.$$

Note that r is a member of the set D_{n+1}. We show that $\varphi(r) = 0$. As in the proof of case II.A we fix $k, l \in \mathbb{N} \cup \{0\}$ such that $l q_0^{n+1} + kq = s$. Such a choice is possible since q and q_0 are coprime and $q < 0 < q_0$. Moreover, we have

$$\varphi\left((a_1 a_2^m)^n x_0 - \sum_{i=1}^{n} (a_1 a_2^m)^i b_1 - \sum_{i=0}^{n-1} \sum_{j=1}^{m} a_1^i a_2^{im+j} b_2 - \frac{k b_0}{q_0^n} \right) = 0$$

since the argument from the above equality belongs to D_n. Applying equality (2.10) to the point from the last equality, for every $i \in I$ and $b \in \mathrm{supp}\mu_i$ we get

$$\varphi\left(a_i \left((a_1 a_2^m)^n x_0 - \sum_{i=1}^{n} (a_1 a_2^m)^i b_1 - b - \sum_{i=0}^{n-1} \sum_{j=1}^{m} a_1^i a_2^{im+j} b_2 - \frac{k b_0}{q_0^n} \right) \right) = 0.$$

In particular, for $i = 1$ and $b = b_1$, we have

$$\varphi\left(a_1 (a_1 a_2^m)^n x_0 - a_1 \sum_{i=1}^{n} (a_1 a_2^m)^i b_1 - a_1 b_1 - \sum_{i=0}^{n-1} \sum_{j=1}^{m} a_1^{i+1} a_2^{im+j} b_2 - a_1 \frac{k b_0}{q_0^n} \right) = 0.$$

If we apply Lemma 2.3.1 with $(c_j, d_j) = (a_2, b_2)$, where $j = 1, 2, ..., m$, to the argument from the previous equality and use the identities

$$a_1 a_2^m \sum_{i=1}^{n} (a_1 a_2^m)^i b_1 + a_1 a_2^m b_1 = \sum_{i=1}^{n+1} (a_1 a_2^m)^i b_1$$

and

$$a_2^m \sum_{i=0}^{n-1} \sum_{j=1}^{m} a_1^{i+1} a_2^{im+j} b_2 + \sum_{j=1}^{m} a_2^j b_2 = \sum_{i=0}^{n} \sum_{j=1}^{m} a_1^i a_2^{im+j} b_2,$$

then we get

$$\varphi\left((a_1 a_2^m)^{n+1} x_0 - \sum_{i=1}^{n+1} (a_1 a_2^m)^i b_1 - \sum_{i=0}^{n} \sum_{j=1}^{m} a_1^i a_2^{im+j} b_2 - a_1 a_2^m \frac{k b_0}{q_0^n} \right) = 0.$$

Since $a_1 a_2^m = q/q_0$, the last equality can be rewritten as

$$\varphi\left((a_1 a_2^m)^{n+1} x_0 - \sum_{i=1}^{n+1} (a_1 a_2^m)^i b_1 - \sum_{i=0}^{n} \sum_{j=1}^{m} a_1^i a_2^{im+j} b_2 - \frac{q k b_0}{q_0^{n+1}} \right) = 0.$$

Remember that $a_0 = 1$. If we use Lemma 2.3.1 with $(c_j, d_j) = (a_0, b_0)$, where $j = 1, 2, ..., l$, to the number from the above equality, then we will get

$$\begin{aligned} 0 =& \varphi\left((a_1 a_2^m)^{n+1} x_0 - \sum_{i=1}^{n+1} (a_1 a_2^m)^i b_1 - \sum_{i=0}^{n} \sum_{j=1}^{m} a_1^i a_2^{im+j} b_2 - \frac{q k b_0}{q_0^{n+1}} - l b_0 \right) \\ =& \varphi\left((a_1 a_2^m)^{n+1} x_0 - \sum_{i=1}^{n+1} (a_1 a_2^m)^i b_1 - \sum_{i=0}^{n} \sum_{j=1}^{m} a_1^i a_2^{im+j} b_2 - \frac{l q_0^{n+1} + k q}{q_0^{n+1}} b_0 \right). \end{aligned}$$

We have chosen the numbers k and l in such a way that $l q_0^{n+1} + k q = s$. Hence

$$\varphi\left((a_1 a_2^m)^{n+1} x_0 - \sum_{i=1}^{n+1} (a_1 a_2^m)^i b_1 - \sum_{i=0}^{n} \sum_{j=1}^{m} a_1^i a_2^{im+j} b_2 - \frac{s b_0}{q_0^{n+1}} \right) = 0,$$

i.e. $\varphi(r) = 0$. Since r was taken arbitrarily this equality implies that $\varphi_{|D_{n+1}} = 0$. In view of the mathematical induction we have $\varphi_{|D} = 0$ and the proof is complete in case II.B.

III. We shall consider the last case when supp$\mu \subset \mathbb{Z} \times \mathbb{R}$, that is $A \subseteq \mathbb{Z}$. We assume additionally that φ is bounded and, according to Remark 2.3.1, we can put $\sup \varphi(\mathbb{R}) \leq 1$. Remember that $a_0 = 1$ and $b_0 \neq 0$. Since $\mu\big(((-\infty, 0) \setminus \{-1\}) \times \mathbb{R}\big) > 0$ we can choose $i \in I$ such that a_i is a negative integer different from -1. Without loss of generality we may assume that $i = 1$. Fix any $b_1 \in$ suppμ_1. We define a sequence of sets $(D_n)_{n \in \mathbb{N} \cup \{0\}}$ setting

$$D_n := \Big\{ e_p ... e_1 x_0 - \sum_{i=1}^{p} e_p ... e_i f_i - \frac{s b_0}{a_1^n} : p \in \mathbb{N}, s \in \mathbb{Z}, (e_1, f_1), ..., (e_p, f_p) \in \text{supp}\mu$$
$$\text{and } e_{i_0} = a_1 \text{ for some } i_0 \in \{1, 2, ..., p\} \Big\},$$

and let

$$D := \bigcup_{n=0}^{\infty} D_n.$$

Note that D_n contains the set

$$\Big\{ a_1(x_0 - b_1) - \frac{s b_0}{a_1^n} : \quad s \in \mathbb{Z} \Big\}$$

as a subset. We have $b_0 \neq 0$ and $a_1^2 > 1$. Lemma 2.3.2 asserts that the set

$$\bigcup_{n=0}^{\infty} \Big\{ a_1(x_0 - b_1) - \frac{s b_0}{a_1^{2n}} : \quad s \in \mathbb{Z} \Big\}$$

is dense in $\mathbb{R}$. Hence also D is dense. Moreover, now we will show that the sequence $(D_n)_{n \in \mathbb{N} \cup \{0\}}$ has the following properties:

P1 If $r \in D_n$ for some $n \in \mathbb{N}$ and $i \in I \setminus \{1\}$, then

$$a_i(r - b) \in D_n \quad \text{for every } b \in \text{supp}\mu_i.$$

P2 If $r \in D_n$ for some $n \in \mathbb{N}$, then

$$a_1(r - b) \in D_{n-1} \quad \text{for every } b \in \text{supp}\mu_1.$$

Fix $n \in \mathbb{N}$ and take any $r \in D_n$. Let $p \in \mathbb{N}$ and $(e_1, f_1), ..., (e_p, f_p) \in$ suppμ be such that a_1 is one of $e_1, ..., e_p$ and

$$r = e_p ... e_1 x_0 - \sum_{i=1}^{p} e_p ... e_i f_i - \frac{s b_0}{a_1^n}$$

with some $s \in \mathbb{Z}$. Take an arbitrary $i \in I$ and $b \in \mathrm{supp}\mu_i$. Then $(a_i, b) \in \mathrm{supp}\mu$ and

$$a_i(r-b) = a_i e_p ... e_1 x_0 - a_i \sum_{i=1}^{p} e_p ... e_i f_i - a_i b - \frac{a_i s b_0}{a_1^n},$$

that is

$$a_i(r-b) = e_{p+1} e_p ... e_1 x_0 - \sum_{i=1}^{p+1} e_{p+1} ... e_i f_i - \frac{a_i s b_0}{a_1^n},$$

where $e_{p+1} := a_i$ and $f_{p+1} := b$. Since $A \subseteq \mathbb{Z}$ we have $a_i s \in \mathbb{Z}$. Consequently, $a_i(r-b) \in D_n$. If, in addition, $i = 1$, then $a_i(r-b) = a_1(r-b) \in D_{n-1}$.

Now we are going to show that the function φ is constant on D. As before we will use the mathematical induction and start with showing that $\varphi_{|D_0} = 0$. Let us remind that

$$D_0 = \Big\{ e_p ... e_1 x_0 - \sum_{i=1}^{p} e_p ... e_i f_i - s b_0 : \ p \in \mathbb{N}, s \in \mathbb{Z}, (e_1, f_1), ..., (e_p, f_p) \in \mathrm{supp}\mu$$
$$\text{and } e_{i_0} = a_1 \text{ for some } i_0 \in \{1, 2, ..., p\} \Big\}.$$

Fix any $p \in \mathbb{N}$, $s \in \mathbb{Z}$ and $(e_1, f_1), ..., (e_p, f_p) \in \mathrm{supp}\mu$ and assume that $e_{i_0} = a_1$ for some $i_0 \in \{1, 2, ..., p\}$. In the first step we show the equality

$$\varphi\Big(e_{i_0} ... e_1 x_0 - \sum_{i=1}^{i_0} e_{i_0} ... e_i f_i - s b_0 \Big) = 0. \tag{2.12}$$

Applying Lemma 2.3.1 with $(c_j, d_j) = (e_j, f_j)$, where $j = 1, 2, ..., i_0 - 1$, to the point x_0 we get

$$\varphi\Big(e_{i_0-1} ... e_1 x_0 - \sum_{i=1}^{i_0-1} e_{i_0-1} ... e_i f_i \Big) = 0.$$

Fix arbitrary $k \in \mathbb{N} \cup \{0\}$ and remember that $a_0 = 1$. If we use Lemma 2.3.1 with $(c_j, d_j) = (a_0, b_0)$, where $j = 1, 2, ..., k$, to the point from the above equality, then we obtain

$$\varphi\Big(e_{i_0-1} ... e_1 x_0 - \sum_{i=1}^{i_0-1} e_{i_0-1} ... e_i f_i - k b_0 \Big) = 0.$$

Putting $x = e_{i_0-1}...e_1x_0 - \sum_{i=1}^{i_0-1} e_{i_0-1}...e_if_i - kb_0$ in equality (2.10) we have

$$\varphi\big(a_i(x-b)\big) = 0 \quad \text{for every } i \in I \text{ and } b \in \text{supp}\mu_i.$$

If we put $a_i = e_{i_0}$ and $b = f_{i_0}$ in the last equality, then we come to

$$\varphi\Bigg(e_{i_0}...e_1x_0 - \sum_{i=1}^{i_0} e_{i_0}...e_if_i - e_{i_0}kb_0\Bigg) = 0.$$

Let $l \in \mathbb{N} \cup \{0\}$ be arbitrarily fixed. Using Lemma 2.3.1 with $(c_j, d_j) = (a_0, b_0)$, where $j = 1, 2, ..., l$, to the point from the previous equality we get

$$\varphi\Bigg(e_{i_0}...e_1x_0 - \sum_{i=1}^{i_0} e_{i_0}...e_if_i - (l + e_{i_0}k)b_0\Bigg) = 0.$$

Since e_{i_0} is a negative integer and $k, l \in \mathbb{N} \cup \{0\}$, the expression $l + e_{i_0}k$ can attain any integer value. In particular, we can find $k, l \in \mathbb{N} \cup \{0\}$ such that $l + e_{i_0}k = s$. This means that equality (2.12) holds.

We use equality (2.12) to prove that

$$\varphi\Bigg(e_p...e_1x_0 - \sum_{i=1}^{p} e_p...e_if_i - sb_0\Bigg) = 0. \tag{2.13}$$

First note that in the case $i_0 = p$ equality (2.12) reduces to (2.13). Therefore we assume that $i_0 < p$. Since $e_{i_0+1}, ..., e_p$ are integers, we can find $\tilde{s} \in \mathbb{Z}$ and $k \in \mathbb{N} \cup \{0\}$ such that $s = k + e_p...e_{i_0+1}\tilde{s}$. If we use Lemma 2.3.1 with $(c_j, d_j) = (e_{i_0+j}, f_{i_0+j})$, where $j = 1, 2, ..., p - i_0$, to the point $e_{i_0}...e_1x_0 - \sum_{i=1}^{i_0} e_{i_0}...e_1f_i - \tilde{s}b_0$, then we will get

$$\varphi\Bigg(e_p...e_1x_0 - \sum_{i=1}^{p} e_p...e_if_i - e_p...e_{i_0+1}\tilde{s}b_0\Bigg) = 0.$$

Applying Lemma 2.3.1 with $(c_j, d_j) = (a_0, b_0)$, where $j = 1, 2, ..., k$, to the point from the last equality we come to

$$\varphi\Bigg(e_p...e_1x_0 - \sum_{i=1}^{p} e_p...e_if_i - (k + e_p...e_{i_0+1}\tilde{s})b_0\Bigg) = 0.$$

Putting $s = k + e_p ... e_{i_0+1}\tilde{s}$, we get

$$\varphi\Bigg(e_p...e_1 x_0 - \sum_{i=1}^{p} e_p...e_i f_i - s b_0\Bigg) = 0.$$

Hence $\varphi_{|D_0} = 0$.

Now assume that $\varphi_{|D_n} = 0$ for some $n \in \mathbb{N} \cup \{0\}$. We prove that $\varphi_{|D_{n+1}} = 0$. Let $r \in D_{n+1}$ be fixed and put $J := I \setminus \{1\}$. Property P2 asserts that $a_1(r - b) \in D_n$ for every $b \in \operatorname{supp}\mu_1$. Hence using equality (2.10) and the inductive hypothesis we get

$$\begin{aligned}
\varphi(r) &= \sum_{i \in I} \int_{\mathbb{R}} \varphi\big(a_i(r-b)\big)\mu_i(db) \\
&= \sum_{i \in J} \int_{\mathbb{R}} \varphi\big(a_i(r-b)\big)\mu_i(db) + \int_{\mathbb{R}} \varphi\big(a_1(r-b)\big)\mu_1(db) \\
&= \sum_{i \in J} \int_{\mathbb{R}} \varphi\big(a_i(r-b)\big)\mu_i(db) + 0 = \sum_{i \in J} \int_{\mathbb{R}} \varphi\big(a_i(r-b)\big)\mu_i(db).
\end{aligned}$$

The function φ is bounded above by 1. Therefore

$$\varphi(r) = \sum_{i \in J} \int_{\mathbb{R}} \varphi\big(a_i(r-b)\big)\mu_i(db) \le \sum_{i \in J} \int_{\mathbb{R}} \mu_i(db) = \sum_{i \in J} \mu_i(\mathbb{R}) = 1 - \mu_1(\mathbb{R}).$$

If we use equation (2.10) for every $i \in J$, then we will get

$$\int_{\mathbb{R}} \varphi(a_i(r-b_1))\mu_i(db_1) = \int_{\mathbb{R}} \left[\sum_{j \in I} \int_{\mathbb{R}} \varphi\Bigg(a_j\big(a_i(r-b_1) - b_2\big)\Bigg)\mu_j(db_2) \right] \mu_i(db_1).$$

Property P1 asserts that $a_i(r - b_1) \in D_{n+1}$ for every $i \in J$ and $b_1 \in \operatorname{supp}\mu_i$. Moreover, property P2 implies that $a_1\big(a_i(r - b_1) - b_2\big) \in D_n$ for every $i \in J$ and $(a_i, b_1), (a_1, b_2) \in \operatorname{supp}\mu$. Since $\varphi_{|D_n} = 0$, we have

$$\varphi\big(a_1\big(a_i(r-b_1) - b_2\big)\big) = 0 \quad \text{for every } i \in J \text{ and } (a_i, b_1), (a_1, b_2) \in \operatorname{supp}\mu.$$

Hence and from the fact that φ is bounded above by 1 we have for all $i \in I$ the inequalities

$$\begin{aligned}\int\limits_{\mathbb{R}} \varphi(a_i(r-b_1))\mu_i(db_1) =& \int\limits_{\mathbb{R}} \left[\sum_{j\in I} \int\limits_{\mathbb{R}} \varphi\Big(a_j\big(a_i(r-b_1)-b_2\big)\Big)\mu_j(db_2)\right]\mu_i(db_1)\\ =& \int\limits_{\mathbb{R}} \left[\sum_{j\in J} \int\limits_{\mathbb{R}} \varphi\Big(a_j\big(a_i(r-b_1)-b_2\big)\Big)\mu_j(db_2)\right]\mu_i(db_1)\\ \leq& \int\limits_{\mathbb{R}} \left[\sum_{j\in J} \int\limits_{\mathbb{R}} \mu_j(db_2)\right]\mu_i(db_1) = \int\limits_{\mathbb{R}} \left[\sum_{j\in J} \mu_j(\mathbb{R})\right]\mu_i(db_1)\\ =& \mu_i(\mathbb{R})\sum_{j\in J}\mu_j(\mathbb{R}) = \mu_i(\mathbb{R})(1-\mu_1(\mathbb{R})).\end{aligned}$$

If we put the above estimations into the equality

$$\varphi(r) = \sum_{i\in J}\int_{\mathbb{R}} \varphi\big(a_i(r-b)\big)\mu_i(db),$$

then we will get

$$\varphi(r) \leq \big(1-\mu_1(\mathbb{R})\big)\sum_{i\in J}\mu_i(\mathbb{R}) = \big(1-\mu_1(\mathbb{R})\big)^2.$$

In a similar way, using several times P1 and P2 and taking into account that the function φ vanishes on D_n and fact that φ satisfies equation (2.10), one can inductively show that

$$\varphi(r) = \sum_{i_1,\dots,i_q\in J}\int_{\mathbb{R}^q} \varphi\Big(a_{i_q}\big(\dots(a_{i_1}(r-b_1)-b_2)\dots-b_{i_q}\big)\Big)\big(\mu_{i_1}\otimes\dots\otimes\mu_{i_q}\big)(db_1,\dots,db_q),$$

where $q\in\mathbb{N}$. Therefore, since all values of φ lie in $[0,1]$, we have

$$\varphi(r) \leq \big(1-\mu_1(\mathbb{R})\big)^q \quad \text{for every } q\in\mathbb{N},$$

and thus $\varphi(r) = 0$ because of the condition $\mu_1(\mathbb{R}) \in (0,1)$. Consequently, we get $\varphi(x) = 0$ for every $x \in D_{n+1}$. Summarizing we see that φ vanishes on D. □

Remark 2.3.2. The assertion of Theorem 2.3.1 is false when $\mu\big(\{-1,1\}\times\mathbb{R}\big) = 1$. This case is covered by Theorem 2.2.2.

The condition $(1, b) \in \text{supp}\mu$, where $b \neq 0$, is not satisfied for all measures obviously. Thus, I was trying to find another result like Theorem 2.3.1. I show the effect of my work below. We start with a very simple fact.

Lemma 2.3.3 (Lemma 2.8, [37]). *Let $a \in (-1, 1) \setminus \{0\}$ and $t \in (0, \infty)$. If $(u_n)_{n\in\mathbb{N}}$ is a sequence of negative numbers, then the set*

$$\bigcup_{n=1}^{\infty} \{u_n + a^n kt : \quad k \in \mathbb{N} \cup \{0\}\}$$

is dense in the positive half line.

Proof. Let $x, \varepsilon \in (0, \infty)$ be fixed. Choose $n_0 \in \mathbb{N}$ such that $a^{2n_0} t < \varepsilon$ and take any $n \geq n_0$. Note that $a^{2n} > 0$, thus

$$(0, \infty) \subset \bigcup_{k=0}^{\infty} \big[u_{2n} + a^{2n} kt, u_{2n} + a^{2n}(k + 1)t\big).$$

We can find $k_0 \in \mathbb{N} \cup \{0\}$ such that $x \in \big[u_{2n} + a^{2n} k_0 t, u_{2n} + a^{2n}(k_0 + 1)t\big)$. In the end observe that the length of this interval is less then ε. □

We are going to prove the next theorem giving another conditions under which every continuous solution of the archetypal equation attaining the global extremum is constant.

Theorem 2.3.2 (Theorem 2.9, [37]). *Assume that $\mu\big((-\infty, 0) \times \mathbb{R}\big) > 0$ and there exist points $(a_1, b_1), ..., (a_s, b_s) \in \text{supp}\mu$ such that $\min\{|a_1|, ..., |a_s|\} < 1$, $a_1 \cdot ... \cdot a_s = -1$ and*

$$\sum_{i=1}^{2s} a_{\sigma(2s)} \cdot ... \cdot a_{\sigma(i)} b_{\sigma(i)} \neq 0, \tag{2.14}$$

for some integer $s \geq 2$ and $\sigma : \{1, 2, ..., 2s\} \to \{1, 2, ..., s\}$ which takes each value exactly twice, then every continuous solution $\varphi : \mathbb{R} \to \mathbb{R}$ of equation (2.1) attaining the global extremum is constant.

Proof. According to Remark 2.3.1 we may assume that $\inf \varphi(\mathbb{R}) = 0$ and there exists $x_0 \in \mathbb{R}$ such that $\varphi(x_0) = 0$. Observe that the assumptions imposed on $a_1, ..., a_s$ and σ give

$$a_{\sigma(1)} \cdot ... \cdot a_{\sigma(2s)} = (a_1 \cdot ... \cdot a_s)^2 = (-1)^2 = 1.$$

If we apply Lemma 2.3.1 with $(c_i, d_i) = (a_{\sigma(i)}, b_{\sigma(i)})$ for $i = 1, ..., 2s$ to the point x_0, then we get

$$\varphi\Bigg(a_{\sigma(1)}...a_{\sigma(2s)}x_0 - \sum_{i=1}^{2s} a_{\sigma(2s)}...a_{\sigma(i)}b_{\sigma(i)}\Bigg) = \varphi\Bigg(x_0 - \sum_{i=1}^{2s} a_{\sigma(2s)}...a_{\sigma(i)}b_{\sigma(i)}\Bigg) = 0.$$

Denote $-\sum_{i=1}^{2s} a_{\sigma(2s)} \cdot ... \cdot a_{\sigma(i)}b_{\sigma(i)}$ by t for simplicity. Then the last equality can be rewritten as

$$\varphi(x_0 + t) = 0.$$

If we again use Lemma 2.3.1 with $(c_i, d_i) = (a_{\sigma(i)}, b_{\sigma(i)})$ for $i = 1, ..., 2s$ to the point $x_0 + t$, then we get $\varphi(x_0 + 2t) = 0$. By the induction one can prove that

$$\varphi\big(x_0 + kt\big) = 0 \quad \text{for every } k \in \mathbb{N} \cup \{0\}. \tag{2.15}$$

Now fix any $k \in \mathbb{N} \cup \{0\}$. Using Lemma 2.3.1 with $(c_i, d_i) = (a_i, b_i)$ for every $i = 1, 2, ..., s$ to the point $x_0 + kt$, then we come to

$$\varphi\Bigg(a_s \cdot ... \cdot a_1(x_0 + kt) - \sum_{i=1}^{s} a_s \cdot ... \cdot a_i b_i\Bigg) = \varphi\Bigg(-x_0 - kt - \sum_{i=1}^{s} a_s \cdot ... \cdot a_i b_i\Bigg) = 0$$

since $a_1 \cdot ... \cdot a_s = -1$. Therefore we have

$$\varphi\big(y_0 - kt\big) = 0 \quad \text{for all } k \in \mathbb{N} \cup \{0\}, \tag{2.16}$$

where $y_0 := -x_0 - \sum_{i=1}^{s} a_s...a_i b_i$.

Without loss of generality we may assume that $|a_1| < 1$. Let $k \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$ be taken arbitrarily. If we use Lemma 2.3.1 with $(c_i, d_i) = (a_1, b_1)$ for every $i = 1, 2, ..., n$ to the points $x_0 + kt$ and $y_0 - kt$, then we obtain the equalities

$$\varphi\Bigg(a_1^n x_0 - \sum_{j=1}^{n} a_1^j b_1 + a_1^n kt\Bigg) = 0 \tag{2.17}$$

and

$$\varphi\Bigg(a_1^n y_0 - \sum_{j=1}^{n} a_1^j b_1 - a_1^n kt\Bigg) = 0, \tag{2.18}$$

respectively.

We know that $t \neq 0$, that is either positive, or negative. We consider the first case. Conditions (2.15) and (2.16) imply that the set of zeros of the function φ

is unbounded both from above and from below. Therefore for every $n \in \mathbb{N}$ we can choose a zero $u_n \in \mathbb{R}$ of the function φ such that

$$a_1^n u_n - \sum_{j=1}^{n} a_1^j b_1 < 0.$$

We define also a sequence $(v_n)_{n\in\mathbb{N}}$ of zeros of the function φ fulfilling the inequalities

$$a_1^n\Big(- v_n - \sum_{i=1}^{s} a_s ... a_i b_i\Big) - \sum_{j=1}^{n} a_1^j b_1 > 0 \quad \text{for every } n \in \mathbb{N}.$$

Equalities (2.17) and (2.18) imply that for every $n \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{0\}$ we have

$$\varphi\Big(a_1^n u_n - \sum_{j=1}^{n} a_1^j b_1 + a_1^n kt \Big) = 0$$

and

$$\varphi\Big(a_1^n\Big(- v_n - \sum_{i=1}^{s} a_s ... a_i b_i\Big) - \sum_{j=1}^{n} a_1^j b_1 - a_1^n kt \Big) = 0.$$

Put

$$E := \bigcup_{n=1}^{\infty} \Big\{ a_1^n u_n - \sum_{j=1}^{n} a_1^j b_1 + a_1^n kt : \quad k \in \mathbb{N} \cup \{0\} \Big\}$$

and

$$F := \bigcup_{n=1}^{\infty} \Big\{ a_1^n\Big(- v_n - \sum_{i=1}^{s} a_s ... a_i b_i\Big) - \sum_{j=1}^{n} a_1^j b_1 - a_1^n kt : \quad k \in \mathbb{N} \cup \{0\} \Big\}.$$

Then $\varphi_{|E\cup F} = 0$. Lemma 2.3.3 asserts that E and $-F$ are dense in $(0, \infty)$. Hence $E \cup F$ is dense in $\mathbb{R}$, and thus φ is constant. If t is negative, the proof is similar and we omit it. $\square$

If $s = 2$ condition (2.14) reduces to a simpler form which is easy to check.

Corollary 2.3.1 (Corollary 2.10, [37]). *Assume that* $\mu\big((-\infty, 0) \times \mathbb{R}\big) > 0$ *and there exist* $(a_1, b_1), (a_2, b_2) \in \operatorname{supp}\mu$ *such that* $|a_1| \neq 1$, $a_1 a_2 = -1$ *and*

$$b_2 \neq \frac{a_1^2 + a_1}{a_1 - 1} b_1.$$

Then every continuous solution of equation (2.1) *attaining the global extremum is constant.*

Proof. We shall check that all assumptions of Theorem 2.3.2 are satisfied. Obviously conditions $|a_1| \neq 1$ and $a_1 a_2 = -1$ imply that either $|a_1| < 1$ or $|a_2| < 1$. It remains to check (2.14). We can treat σ as a sequence $(\sigma(1), \sigma(2), \sigma(3), \sigma(4))$ of digits 1 and 2; each of them appears twice. We will see that for $\sigma = (1,1,2,2)$ condition (2.14) holds. Putting equality $a_2 = -\frac{1}{a_1}$ into the sum $\sum_{i=1}^{2s} a_{\sigma(2s)} \cdot \ldots \cdot a_{\sigma(i)} b_{\sigma(i)}$ we get

$$a_1^2 a_2^2 b_1 + a_1 a_2^2 b_1 + a_2^2 b_2 + a_2 b_2 = b_1 + \frac{1}{a_1} b_1 + \frac{1}{a_1^2} b_2 - \frac{1}{a_1} b_2.$$

The above expression takes value 0 if and only if

$$\left(1 + \frac{1}{a_1}\right) b_1 = \left(\frac{1}{a_1} - \frac{1}{a_1^2}\right) b_2,$$

that is

$$b_2 = \frac{a_1^2 + a_1}{a_1 - 1} b_1.$$

Since we assumed that $b_2 \neq \frac{a_1^2 + a_1}{a_1 - 1} b_1$, condition (2.14) is satisfied. □

Unfortunately the class of measures from Theorems 2.3.1 and 2.3.2 still do not cover all possibilities. This fact can be illustrated by the simple example from [37].

Example 2.3.1. Take any probability measure μ on $\mathbb{R}^2$ with the support $\{(-2,1),(2,2)\}$. Then we cannot use Theorem 2.3.1 since there is no point of the form $(1,b)$ with $b \neq 0$ in suppμ. We cannot use also Theorem 2.3.2 as suppμ does not contain a point of the form $(a,b) \in \mathbb{R}^2$ with $|a| < 1$.

Other examples will be presented at the end of the next section.

2.4 Oscillating solutions

In this part we consider solutions which do not attain the global extremum. The below analyses have been made by me in [37]. We begin with the following

Theorem 2.4.1 (Theorem 3.1, [37]). *Assume that* $\mu\big(((-\infty,0)\setminus\{-1\})\times\mathbb{R}\big) > 0$ *and there exists* $b \in \mathbb{R}\setminus\{0\}$ *such that* $(1,b) \in \operatorname{supp}\mu$. *Let* $\varphi : \mathbb{R} \to \mathbb{R}$ *be a bounded continuous solution of* (2.1). *Then*

$$\liminf_{x\to-\infty} \varphi(x) = \liminf_{x\to\infty} \varphi(x) = \inf \varphi(\mathbb{R}) \tag{2.19}$$

and

$$\limsup_{x\to-\infty} \varphi(x) = \limsup_{x\to\infty} \varphi(x) = \sup \varphi(\mathbb{R}). \tag{2.20}$$

Proof. We will show equalities (2.19) only. Choose a sequence $(x_n)_{n\in\mathbb{N}}$ of reals such that

$$\varphi(x_n) \to \inf \varphi(\mathbb{R}).$$

If $(x_n)_{n\in\mathbb{N}}$ is bounded, then one can find its subsequence $(y_n)_{n\in\mathbb{N}}$ which is convergent to some $x_0 \in \mathbb{R}$. Then, by the continuity of the function φ, we have $\varphi(x_0) = \inf \varphi(\mathbb{R})$. Hence, by Theorem 2.3.1, we know that φ is constant and equalities (2.19) hold. So we may assume that $(x_n)_{n\in\mathbb{N}}$ is unbounded. Then we can choose its subsequence $(y_n)_{n\in\mathbb{N}}$ such that either $y_n \to -\infty$, or $y_n \to \infty$. Consider, for instance, the first possibility. Then

$$\lim_{n\to\infty} \varphi(y_n) = \inf \varphi(\mathbb{R}),$$

and thus

$$\liminf_{x\to-\infty} \varphi(x) = \inf \varphi(\mathbb{R}).$$

Since, in view of Theorem 4.2 from [10], we have

$$\liminf_{x\to-\infty} \varphi(x) = \liminf_{x\to\infty} \varphi(x),$$

we come to (2.19). □

The following result can be deduced immediately from the above theorem.

Corollary 2.4.1 (Corollary 3.2, [37]). *Let* $\mu\big(((-\infty,0)\setminus\{-1\})\times\mathbb{R}\big) > 0$ *and let* $b \in \mathbb{R}\setminus\{0\}$ *be such that* $(1,b) \in \operatorname{supp}\mu$. *Assume that* $\varphi : \mathbb{R} \to \mathbb{R}$ *is a bounded continuous solution of* (2.1). *If at least one of the limits* $\lim_{x\to-\infty} \varphi(x)$ *and* $\lim_{x\to\infty} \varphi(x)$ *exists, then* φ *is constant.*

Corollary 2.4.1 and Theorem 2.3.1 provide a partial solution to the problem posed by Gregory Derfel. More precisely, we have

Corollary 2.4.2. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a bounded continuous solution of equation* (2.8)*. If the function φ attains its global extremum or there exists at least one of the limits $\lim_{x\to-\infty} \varphi(x)$ and $\lim_{x\to\infty} \varphi(x)$, then φ is constant.*

We have also the analogical results generated by Theorem 2.3.2.

Theorem 2.4.2 (Theorem 3.3, [37])**.** *Assume that $\mu\big((-\infty, 0) \times \mathbb{R}\big) > 0$ and there exist points $(a_1, b_1), ..., (a_s, b_s) \in \mathrm{supp}\mu$ such that $\min\{|a_1|, ..., |a_s|\} < 1$, $a_1 \cdot ... \cdot a_s = -1$ and condition* (2.14) *is satisfied for some integer $s \geq 2$ and $\sigma : \{1, 2, ..., 2s\} \to \{1, 2, ..., s\}$ taking the values exactly twice. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a bounded continuous solution of* (2.1)*. Then*

$$\liminf_{x\to-\infty} \varphi(x) = \liminf_{x\to\infty} \varphi(x) = \inf \varphi(\mathbb{R})$$

and

$$\limsup_{x\to-\infty} \varphi(x) = \limsup_{x\to\infty} \varphi(x) = \sup \varphi(\mathbb{R}).$$

Corollary 2.4.3 (Corollary 3.4, [37])**.** *Assume that $\mu\big((-\infty, 0) \times \mathbb{R}\big) > 0$ and there exist points $(a_1, b_1), ..., (a_s, b_s) \in \mathrm{supp}\mu$ such that $\min\{|a_1|, ..., |a_s|\} < 1$, $a_1 \cdot ... \cdot a_s = -1$ and condition* (2.14) *is satisfied for some integer $s \geq 2$ and $\sigma : \{1, 2, ..., 2s\} \to \{1, 2, ..., s\}$ taking the values exactly twice. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a bounded continuous solution of* (2.1)*. If at least one of the limits $\lim_{x\to-\infty} \varphi(x)$ and $\lim_{x\to\infty} \varphi(x)$ exists, then φ is constant.*

The above corollaries can be compared with a similar theorem by L. Bogachev, G. Derfel and S. Molchanov (see [10, Theorem 4.3]).

Theorem 2.4.3 (Bogachev, Derfel, Molchanov, [10])**.** *Assume the conditions: $\mu((-\infty, 0)\times\mathbb{R}) > 0$, $\mu(\{0\}\times\mathbb{R}) = 0$, $\mu(\{-1, 1\}\times\mathbb{R}) < 1$, $\mu\big(\{(a, b) \in \mathbb{R}^2 : a(c-b) = c\}\big) < 1$ for all $c \in \mathbb{R}$ and assume that $\iint_{\mathbb{R}^2} \ln(\max(|b|, 1))\mu(da, db) < \infty$ and $K = \iint_{\mathbb{R}^2} \ln |a|\mu(da, db) \in (0, \infty)$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a bounded continuous solution of* (2.1)*. If at least one of the limits $\lim_{x\to-\infty} \varphi(x)$ and $\lim_{x\to\infty} \varphi(x)$ exists, then φ is constant.*

The above results (Corollary 2.4.1, Corollary 2.4.3 and Theorem 2.4.3) are not comparable: none of them implies the other. This fact is illustrated by the below examples coming from [37].

Example 2.4.1. Assume that $\mu\big(\{(1,1)\}\big) = 1/2$ and

$$\mu\Bigg(\{(-\mathrm{e}^{2^n},0)\}\Bigg) = \frac{1}{2^n} \quad \text{for } n = 2,3,...$$

Note that in the case like this

$$K = \frac{1}{2}\ln|1| + \sum_{n=2}^{\infty}\frac{1}{2^n}\ln\Bigg| -\mathrm{e}^{2^n}\Bigg| = \sum_{n=2}^{\infty} 1 = \infty.$$

This implies that Theorem 2.4.3 cannot be used. Moreover $\mu\Big(\big((-\infty,-1)\cup(-1,0)\big)\times\mathbb{R}\Big) = 1/2$. Corollary 2.4.1 can be applied because of the inequalities $\mu\Big(\big((-\infty,-1)\cup(-1,0)\big)\times\mathbb{R}\Big) > 0$ and $\mu(\{(1,1)\}) > 0$. Thus each bounded continuous solution $\varphi : \mathbb{R}\to\mathbb{R}$ of the equation

$$\varphi(x) = \frac{1}{2}\varphi(x-1) + \sum_{n=2}^{\infty}\frac{1}{2^n}\varphi(-\mathrm{e}^{2^n}x)$$

having at least one of the limits $\lim_{x\to-\infty}\varphi(x)$ and $\lim_{x\to\infty}\varphi(x)$ is constant.

If we change the above example only in one place, then we can apply neither Corollaries 2.4.1 and 2.4.3, nor Theorem 2.4.3.

Example 2.4.2. Assume that $\mu\big(\{(\mathrm{e},1)\}\big) = 1/2$ and

$$\mu\big(\{(-\mathrm{e}^{2^n},0)\}\big) = \frac{1}{2^n} \quad \text{for } n = 2,3,...$$

For the measure μ we cannot use Corollary 2.4.1 since there is no $b\in\mathbb{R}\setminus\{0\}$ such that $\mu\big(\{(1,b)\}\big) > 0$. Further, observe that if $\mu\big(\{(a,b)\}\big) > 0$, then $|a| > 1$. Therefore we cannot apply Corollary 2.4.3 in this example. Theorem 2.4.3 also cannot be used here from the same reason as in the previous example. In consequence, we know nothing about non-constant bounded continuous solutions $\varphi : \mathbb{R}\to\mathbb{R}$ of the equation

$$\varphi(x) = \frac{1}{2}\varphi(\mathrm{e}x-\mathrm{e}) + \sum_{n=2}^{\infty}\frac{1}{2^n}\varphi(-\mathrm{e}^{2^n}x).$$

In the last example Theorem 2.4.3 can be applied but Corollaries 2.4.1 and 2.4.3 cannot.

Example 2.4.3. Assume that $\mu\big(\{(-1,1)\}\big) = \mu\big(\{(-\mathrm{e},0)\}\big) = 1/2$. All assumptions from Theorem 2.4.3 are satisfied. Indeed

$$K = \frac{1}{2}\ln|-1| + \frac{1}{2}\ln|-\mathrm{e}| = \frac{1}{2} < \infty$$

and

$$\iint_{\mathbb{R}^2} \ln(\max(|b|,1))\mu(da,db) = \frac{1}{2}\ln(\max(|1|,1)) + \frac{1}{2}\ln(\max(|\mathrm{e}|,1)) = \frac{1}{2} < \infty.$$

Corollaries 2.4.1 and 2.4.3 cannot be used here from the same reasons as in the previous example. Therefore, in view of Theorem 2.4.3, every bounded continuous solution $\varphi : \mathbb{R} \to \mathbb{R}$ of the equation

$$\varphi(x) = \frac{1}{2}\varphi(-x-1) + \frac{1}{2}\varphi(-\mathrm{e}x),$$

which has at least one of the limits $\lim_{x\to-\infty}\varphi(x)$ and $\lim_{x\to\infty}\varphi(x)$, must be constant.

Chapter 3

Equations with not necessarily affine transforms of the argument

The present chapter contains results which have never been published and submitted as a paper. I decided to present them in the PhD thesis since they are very close to the theorems obtained by me in the previous chapter. During my visit at the Silesian University in Katowice in October 2017, Professor Janusz Morawiec paid my attention to the Kuratowski relation (see Definition 1.1.1). As we will see this equivalence relation is strictly connected with functional equations in a single variable. When I was a guest of Janusz Morawiec I had already known that any bounded continuous solution of equation (2.8) attaining its global extremum must be constant. I generalized this result in a few directions as we could see in Chapter 2, see also [36]. I decided to examine functional equations for which transforms of arguments are arbitrary homeomorphisms of the real line. For this purpose I introduced the *compatibility conditions* – for their definitions see Section 3.1.

In my opinion, the Kuratowski relation can be very helpful to deeper understanding the problem posed by Gregory Derfel and why his equation (2.8) is so hard to solve – a dicussion on the connection between the Kuratowski equivalence relation and this equation the reader will find in Chapter 5.

In this chapter we fix a set $I \subseteq \mathbb{Z}$ and the family $\mathcal{F} = \{f_i : \mathbb{R} \to \mathbb{R} | \quad i \in I\}$ of homeomorphisms. We have given also a set of positive reals $\{p_i : i \in I\}$ summing up to 1. In the present chapter we shall examine the equation

$$\varphi(x) = \sum_{i \in I} p_i \varphi\big(f_i(x)\big). \tag{3.1}$$

We are looking for its continuous solutions $\varphi : \mathbb{R} \to \mathbb{R}$.

Chapter 3 contains three sections. In the first one we introduce the compatibility conditions and examine relations between them. Section 4.2 is devoted to solutions of equation (3.1) attaining the global extremum on the equivalence class generated by the Kuratowski relation. That section contains also results connected with continuous solutions of (3.1). In the last section we examine solutions having the limits at ∞ and $-\infty$ and present one example illustrating our analysis.

It is worth adding that all presented results and examples are obtained by me and they have never been written down before. For these reasons the bibliographical details do not appear in statements of lemmas, theorems etc.

3.1 Compatibility conditions

We will analyse equation (3.1) when the inner functions satisfy some compatibility conditions. We define them as follows.

Definition 3.1.1. We say that the family $\{f_i : \mathbb{R} \to \mathbb{R} | \quad i \in I\}$ satisfies the *weak compatibility condition* if for all $i, j \in I$ there exist $k \in \mathbb{N}$ and $i_1, ..., i_k \in I$ such that

$$f_i \circ f_j \circ f_i^{-1} = f_{i_1} \circ ... \circ f_{i_k}.$$

Definition 3.1.2. We say that the family $\{f_i : \mathbb{R} \to \mathbb{R} | \quad i \in I\}$ satisfies the *strong compatibility condition* if for all $i, j \in I$ there exist $k \in \mathbb{N}$ and $i_1, ..., i_k \in I$ such that

$$f_j \circ f_i^{-1} = f_{i_1} \circ ... \circ f_{i_k}.$$

Since iterates of the inner functions usually appear in the formulas of solutions of such equations, the compatibility conditions provide more regularity for them behaviour.

In the below remarks we will see relations between different types of the compatibility conditions.

Remark 3.1.1. First of all note that if the family $\{f_i : \mathbb{R} \to \mathbb{R} | \quad i \in I\}$ satisfies the strong compatiblity condition, then this family satisfies weak one. The next example shows that we cannot reverse this implication.

Example 3.1.1. Put $I = \mathbb{N}$. We define the family $\mathcal{F}$ as follows

$$f_i(x) = x - i \quad \text{for every } i \in I.$$

Observe that for all $i, j \in I$ we have

$$f_i \circ f_j \circ f_i^{-1} = f_j.$$

This equality means that $\mathcal{F}$ fulfills the weak compatibility condition. Moreover, if $i, j \in I$ are such that $j < i$, then we have

$$\big(f_j \circ f_i^{-1}\big)(x) = x + i - j \quad \text{for all } x \in \mathbb{R}.$$

Since $i - j$ is positive, we are not able to express the above function as a finite number of compositions of mappings from the family $\mathcal{F}$. Therefore this family does not satisfy the strong compatibility condition.

In the above example we could easily express the function $f_i \circ f_j \circ f_i^{-1}$ for every $i, j \in I$ as compositions of the functions $f_i, i \in I$, since the above mappings are pairwise commuting. The situation occuring there is a particular case of a more general phenomena.

Remark 3.1.2. *If the functions $f_i, i \in I$, are pairwise commuting, then $\mathcal{F}$ fulfills the weak compatibily condition.*

Further, it is easy to see that we have the following implication.

Remark 3.1.3. *If for every $i \in I$ there exists $j \in I$ such that $f_i = f_j^{-1}$, then the family $\{f_i : \mathbb{R} \to \mathbb{R} |\ \ i \in I\}$ satisfies the strong compatibility condition.*

Remark 3.1.4. It is worth mentioning that there is an extremly rare case when the above condition is satisfied. If $\{f_i : \mathbb{R} \to \mathbb{R} |\ \ i \in I\}$ contains involutions only, then in view of the above remark this family satisfies the strong compatibility condition. We remind that a function $f : \mathbb{R} \to \mathbb{R}$ is called an *involution* if $f^2 = \mathrm{Id}_{\mathbb{R}}$. This condition is equivalent to the facts saying that f is bijective and $f^{-1} = f$. One can note that if an increasing homeomorphism $f : \mathbb{R} \to \mathbb{R}$ is an involution, then f is the identity. Therefore non-trivial continuous involutions mapping $\mathbb{R}$ onto $\mathbb{R}$ are decreasing. For the details and more interesting facts concerning involutions the reader is referred to [26, Chapter 15] or [29].

3.2 Solutions attaining the global extremum

In this section we prove the results which are similar to the theorems from Chapter 2. Remind that the symbol $\sim$ denotes the Kuratowski equivalence relation (with respect to a given family of homeomorphisms of the real line) – this relation was described in Definition 1.1.1. We denote its equivalence class generated by some real number x as $[x]_\sim$.

At the beginning of this section we prove the following lemma (compare with Lemma 2.3.1).

Lemma 3.2.1. *Assume that* $\varphi : \mathbb{R} \to \mathbb{R}$ *is a solution of equation* (3.1) *and there exists* $x_0 \in \mathbb{R}$ *such that*

$$\varphi(x_0) = \inf \varphi\big([x_0]_\sim\big) \tag{3.2}$$

or

$$\varphi(x_0) = \sup \varphi\big([x_0]_\sim\big). \tag{3.3}$$

Then

$$\varphi(x_0) = \varphi\big((f_{j_1} \circ ... \circ f_{j_n})(x_0)\big) \tag{3.4}$$

for every $n \in \mathbb{N}$ *and* $j_1, ..., j_n \in I$.

Proof. We will prove this lemma only in the case when equality (3.2) holds. Fix any $n \in \mathbb{N}$ and $j_1, ..., j_n \in I$. Observe that since $\varphi(x_0) = \inf \varphi([x_0]_\sim)$, then we have

$$\varphi(x_0) \leq \varphi\big(f_i(x_0)\big) \quad \text{for all } i \in I.$$

Hence and from (3.1) we can deduce

$$\varphi(x_0) = \varphi\big(f_i(x_0)\big) \quad \text{for all } i \in I$$

as non-negative numbers p_i, $i \in I$, sum up to 1. In particular, we have the equality $\varphi(x_0) = \varphi\big(f_{j_n}(x_0)\big)$. Repeating this argument $n-1$ times we come to (3.4). □

In the first half of this section we consider the case when the strong compatibility condition is satisfied. This is my first main theorem connected with equation (3.1) and compatibility conditions.

Theorem 3.2.1. *Assume that* $\mathcal{F}$ *satisfies the strong compatibility condition. If* $\varphi : \mathbb{R} \to \mathbb{R}$ *is a solution of equation* (3.1) *and there exists* $x_0 \in \mathbb{R}$ *such that* (3.2) *or* (3.3) *holds, then* $\varphi_{|[x_0]_\sim}$ *is constant.*

Proof. Assume that (3.2) is satisfied. Fix any $y \in [x_0]_\sim$. We have to show that $\varphi(y) = \varphi(x_0)$. By the definition of the relation $\sim$ we can find $k \in \mathbb{N}$, $i_1, ..., i_k \in I$ and $\varepsilon_1, ..., \varepsilon_k \in \{-1, 1\}$ such that

$$y = \left(f_{i_1}^{\varepsilon_1} \circ ... \circ f_{i_k}^{\varepsilon_k}\right)(x_0). \tag{3.5}$$

Note that it is sufficient to prove the equality

$$\varphi(x_0) = \varphi\big(f_{i_k}^{\varepsilon_k}(x_0)\big),$$

and then, repeating $k-1$ times this argument we get (3.5). Note that the case $\varepsilon_k = 1$ is covered by Lemma 3.2.1. It remains to consider the case $\varepsilon_k = -1$ only.

We will show that $\varphi(f_{i_k}^{-1}(x_0)) = \varphi(x_0)$. If we apply equality (3.1) to the point $f_{i_k}^{-1}(x_0)$, then we will get

$$\varphi\big(f_{i_k}^{-1}(x_0)\big) = \sum_{i \in I} p_i \varphi\big(f_i(f_{i_k}^{-1}(x_0))\big).$$

We know that the family $\mathcal{F}$ satisfies the strong compatibilty condition and equality (3.4) holds. These two facts imply

$$\varphi\big(f_i(f_{i_k}^{-1}(x_0))\big) = \varphi(x_0) \quad \text{for all } i \in I \setminus \{i_k\}.$$

For $i = i_k$ the above equality becomes trivial. Therefore

$$\varphi(f_{i_k}^{-1}(x_0)) = \sum_{i \in I} p_i \varphi\big(f_i(f_{i_k}^{-1})(x_0)\big) = \sum_{i \in I} p_i \varphi(x_0) = \varphi(x_0)$$

and the proof has been completed. If we assumed (3.3), then the proof would be similar. □

If we restrict our consideration to the class of continuous functions, we can deduce the following result.

Corollary 3.2.1. *Assume that $\mathcal{F}$ satisfies the strong compatibility condition and $\varphi : \mathbb{R} \to \mathbb{R}$ is a continuous solution of equation* (3.1)*. If there exists $x_0 \in \mathbb{R}$ such that $[x_0]_\sim$ is dense in $\mathbb{R}$ and $\varphi_{|[x_0]_\sim}$ attains the global extremum, then φ is constant.*

Now we are going to analyse equation (3.1) in the case when the family $\mathcal{F}$ satisfies the weak compatibility condition. The second main theorem of this section reads as follows.

Theorem 3.2.2. *Assume that the family $\mathcal{F}$ satisfies the weak compatibility condition. If $\varphi:\mathbb{R}\to\mathbb{R}$ is a solution of equation* (3.1) *and there exists $x_0\in\mathbb{R}$ such that $\varphi_{|[x_0]_\sim}$ is bounded above [below] and equality* (3.2) [(3.3)] *holds, then $\varphi_{|[x_0]_\sim}$ is constant.*

Proof. Assume that equality (3.2) is satisfied and the function $\varphi_{|[x_0]_\sim}$ is bounded above. As in the proof of Theorem 3.2.1 we need only to show the equality

$$\varphi(x_0)=\varphi\big(f_i^{\varepsilon}(x_0)\big)\quad \text{for all } i\in I \text{ and } \varepsilon\in\{-1,1\}.$$

Note that the above equality holds for $\varepsilon=1$ because of Lemma 3.2.1. Therefore we will restrict our reasoning to the case $\varepsilon=-1$ only. Take any $j\in I$. We will show that $\varphi(f_j^{-1}(x_0))=\varphi(x_0)$. Let $M\in(0,\infty)$ be such that

$$\varphi(x)\le M\quad \text{for all } x\in[x_0]_\sim.$$

First of all note that for every $k\in\mathbb{N}$ and $i_1,...,i_k\in I$ we have

$$f_j\circ(f_{i_1}\circ...\circ f_{i_k})\circ f_j^{-1}=(f_j\circ f_{i_1}\circ f_j^{-1})\circ(f_j\circ f_{i_2}\circ f_j^{-1})\circ...\circ(f_j\circ f_{i_k}\circ f_j^{-1}).$$

The above identity, jointly with the weak compatibility condition and Lemma 3.2.1, imply that

$$\varphi\big((f_j\circ f_{i_1}\circ...\circ f_{i_k}\circ f_j^{-1})(x_0)\big)=\varphi(x_0)\quad \text{for all } k\in\mathbb{N} \text{ and } i_1,...,i_k\in I. \tag{3.6}$$

Now we will show inductively (with respect to $n\in\mathbb{N}$) the equality

$$\begin{aligned}\varphi\big(f_j^{-1}(x_0)\big)=&\big[1-(1-p_j)^n\big]\varphi(x_0)\\&+\sum_{i_1,...,i_n\in I\setminus\{j\}}p_{i_1}...p_{i_n}\varphi\big((f_{i_n}\circ...\circ f_{i_1}\circ f_j^{-1})(x_0)\big).\end{aligned}\tag{3.7}$$

Denote the set $I\setminus\{j\}$ by J for simplicity. At the beginning we check that equality (3.7) holds for $n=1$. Applying equality (3.1) to the point $f_j^{-1}(x_0)$ we obtain

$$\begin{aligned}\varphi\big(f_j^{-1}(x_0)\big)=&\sum_{i\in I}p_i\varphi\big(f_i\big(f_j^{-1}(x_0)\big)\big)\\=&p_j\varphi\big(f_j\big(f_j^{-1}(x_0)\big)\big)+\sum_{i\in J}p_i\varphi\big(f_i\big(f_j^{-1}(x_0)\big)\big)\\=&p_j\varphi(x_0)+\sum_{i\in J}p_i\varphi\big(f_i\big(f_j^{-1}(x_0)\big)\big)\\=&\big[1-(1-p_j)\big]\varphi(x_0)+\sum_{i\in J}p_i\varphi\big((f_i\circ f_j^{-1})(x_0)\big).\end{aligned}$$

Now suppose that (3.7) is satisfied for some $n \in \mathbb{N}$ and note that

$$(1-p_j)^n = \left(\sum_{i\in J} p_i\right)^n = \sum_{(i_1,\dots,i_n)\in J^n} p_{i_1}\dots p_{i_n}. \tag{3.8}$$

Then, using (3.6) and (3.8), we obtain

$$\begin{aligned}
\varphi\big(f_j^{-1}(x_0)\big) =& \big[1-(1-p_j)^n\big]\varphi(x_0) \\
&+ \sum_{(i_1,\dots,i_n)\in J^n} p_{i_1}\dots p_{i_n}\varphi\big((f_{i_n}\circ\dots\circ f_{i_1}\circ f_j^{-1})(x_0)\big) \\
=& \big[1-(1-p_j)^n\big]\varphi(x_0) \\
&+ \sum_{(i_1,\dots,i_n,i_{n+1})\in J^n\times I} p_{i_1}\dots p_{i_{n+1}}\varphi\big((f_{i_{n+1}}\circ\dots\circ f_{i_1}\circ f_j^{-1})(x_0)\big) \\
=& \big[1-(1-p_j)^n\big]\varphi(x_0) \\
&+ \sum_{(i_1,\dots,i_n,i_{n+1})\in J^n\times\{j\}} p_{i_1}\dots p_{i_{n+1}}\varphi\big((f_{i_{n+1}}\circ\dots\circ f_{i_1}\circ f_j^{-1})(x_0)\big) \\
&+ \sum_{(i_1,\dots,i_{n+1})\in J^{n+1}} p_{i_1}\dots p_{i_{n+1}}\varphi\big((f_{i_{n+1}}\circ\dots\circ f_{i_1}\circ f_j^{-1})(x_0)\big) \\
=& \big[1-(1-p_j)^n\big]\varphi(x_0) \\
&+ \sum_{(i_1,\dots,i_n)\in J^n} p_{i_1}\dots p_{i_n}p_j\varphi\big((f_j\circ f_{i_n}\circ\dots\circ f_{i_1}\circ f_j^{-1})(x_0)\big) \\
&+ \sum_{(i_1,\dots,i_{n+1})\in J^{n+1}} p_{i_1}\dots p_{i_{n+1}}\varphi\big((f_{i_{n+1}}\circ\dots\circ f_{i_1}\circ f_j^{-1})(x_0)\big) \\
=& \big[1-(1-p_j)^n\big]\varphi(x_0) + p_j\sum_{(i_1,\dots,i_n)\in J^n} p_{i_1}\dots p_{i_n}\varphi(x_0) \\
&+ \sum_{(i_1,\dots,i_{n+1})\in J^{n+1}} p_{i_1}\dots p_{i_{n+1}}\varphi\big((f_{i_{n+1}}\circ\dots\circ f_{i_1}\circ f_j^{-1})(x_0)\big) \\
=& \big[1-(1-p_j)^n\big]\varphi(x_0) + p_j(1-p_j)^n\varphi(x_0) \\
&+ \sum_{(i_1,\dots,i_{n+1})\in J^{n+1}} p_{i_1}\dots p_{i_{n+1}}\varphi\big((f_{i_{n+1}}\circ\dots\circ f_{i_1}\circ f_j^{-1})(x_0)\big) \\
=& \big[1-(1-p_j)^{n+1}\big]\varphi(x_0) \\
&+ \sum_{(i_1,\dots,i_{n+1})\in J^{n+1}} p_{i_1}\dots p_{i_{n+1}}\varphi\big((f_{i_{n+1}}\circ\dots\circ f_{i_1}\circ f_j^{-1})(x_0)\big).
\end{aligned}$$

By the mathematical induction, equality (3.7) holds for every $n \in \mathbb{N}$. We assumed that the function φ is bounded above by M on $[x_0]_\sim$. Using this fact jointly with equalities (3.7) and (3.8) we can get for every $n \in \mathbb{N}$ the following estimation

$$\begin{aligned}\varphi\big(f_j^{-1}(x_0)\big) =& \big[1-(1-p_j)^n\big]\varphi(x_0) + \sum_{i_1,\dots,i_n \in J} p_{i_1}\dots p_{i_n}\varphi\big((f_{i_n}\circ\dots\circ f_{i_1}\circ f_j^{-1})(x_0)\big)\\ \le& \big[1-(1-p_j)^n\big]\varphi(x_0) + \sum_{i_1,\dots,i_n \in J} p_{i_1}\dots p_{i_n} M\\ =& \big[1-(1-p_j)^n\big]\varphi(x_0) + (1-p_j)^n M.\end{aligned}$$

Since $p_j \in (0,1)$, taking $n \to \infty$ in the above inequality we get

$$\varphi(f_j^{-1}(x_0)) \le \varphi(x_0).$$

Hence $\varphi(f_j^{-1}(x_0)) = \varphi(x_0)$ because of (3.2) and the proof is complete. □

The above theorem is a result of my work with the weak compatibility condition.

Remark 3.2.1. We see that Theorem 3.2.2 is very close to Theorem 3.2.1. In the above statement we assumed that the family $\mathcal{F}$ satisfies the weak compatibility condition in contrast to the strong compatibility condition from Theorem 3.2.1. Nevertheless, the price is an additional assumption saying that $\varphi_{|[x_0]_\sim}$ is bounded above or below.

Furthermore, if we restrict ourselves to bounded continuous solutions of equation (3.1), then we can deduce the corollary which is formulated in the similar way to Corollary 3.2.1.

Corollary 3.2.2. *Assume that $\mathcal{F}$ satisfies the weak compatibility condition and $\varphi : \mathbb{R} \to \mathbb{R}$ is a bounded continuous solution of equation* (3.1)*. If there exists $x_0 \in \mathbb{R}$ such that $[x_0]_\sim$ is dense in $\mathbb{R}$ and $\varphi_{|[x_0]_\sim}$ attains the global extremum, then φ is constant.*

3.3 Further results and examples

Until now we were not interested in the type of monotonicity of homeomorphisms from $\mathcal{F}$. As we will see if at least one member of $\mathcal{F}$ is decreasing, then an asymptotical behaviour of the solutions becomes simpler.

We start our considerations with

Proposition 3.3.1. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a bounded solution of equation* (3.1)*. If at least one function from $\mathcal{F}$ is decreasing, then*

$$\liminf_{x\to-\infty} \varphi(x) = \liminf_{x\to\infty} \varphi(x)$$

and

$$\limsup_{x\to-\infty} \varphi(x) = \limsup_{x\to\infty} \varphi(x).$$

Proof. The presented reasoning is the same as in the proof of Theorem 4.2 from [10] but we will present it for the sake of completeness. We prove the first equality only. Define $J \subseteq I$ as follows

$$J = \big\{ i \in I : \quad f_i \text{ is decreasing} \big\}$$

and put $q := \sum_{i\in J} p_i$. Then

$$\begin{aligned}
\liminf_{x\to-\infty} \varphi(x) =& \liminf_{x\to-\infty} \sum_{i\in I} p_i \varphi\big(f_i(x)\big) \geq \sum_{i\in I} p_i \liminf_{x\to-\infty} \varphi\big(f_i(x)\big) \\
=& \sum_{i\in J} p_i \liminf_{x\to-\infty} \varphi\big(f_i(x)\big) + \sum_{i\in I\setminus J} p_i \liminf_{x\to-\infty} \varphi\big(f_i(x)\big) \\
=& \sum_{i\in J} p_i \liminf_{x\to\infty} \varphi(x) + \sum_{i\in I\setminus J} p_i \liminf_{x\to-\infty} \varphi(x) \\
=& q \liminf_{x\to\infty} \varphi(x) + (1-q) \liminf_{x\to-\infty} \varphi(x).
\end{aligned}$$

Since $q > 0$, we have $\liminf_{x\to-\infty} \varphi(x) \geq \liminf_{x\to\infty} \varphi(x)$. We can get the opposite inequality in the analogical way. Therefore we have $\liminf_{x\to-\infty} \varphi(x) = \liminf_{x\to\infty} \varphi(x)$ and the proof has been finished. □

We get immediately

Corollary 3.3.1. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a bounded solution of equation* (3.1)*. Assume that $\mathcal{F}$ contains a decreasing function. If at least one of the limits $\lim_{x\to-\infty} \varphi(x)$ and $\lim_{x\to\infty} \varphi(x)$ exists, then they both exist and*

$$\lim_{x\to-\infty} \varphi(x) = \lim_{x\to\infty} \varphi(x).$$

Proof. It follows directly from Proposition 3.3.1 and from the fact that at least one of the equalities $\liminf_{x\to-\infty}\varphi(x) = \limsup_{x\to-\infty}\varphi(x)$ and $\liminf_{x\to\infty}\varphi(x) = \limsup_{x\to\infty}\varphi(x)$ holds true. □

I suppose that the above results (Proposition 3.3.1 and Corollary 3.3.1) could be known earlier. Assuming the weak compatibility condition we can obtain a stronger assertion than in Proposition 3.3.1 and the next proposition can be compared with Theorem 2.4.1 or Theorem 2.4.2. I have proven the below result in almost the same way as Theorem 2.4.1.

Proposition 3.3.2. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a bounded continuous solution of equation* (3.1)*. Assume that $\mathcal{F}$ satisfies the weak compatibility condition and $\mathcal{F}$ contains a decreasing function. If the equivalence class $[a]_\sim$ is dense in $\mathbb{R}$ for every $a \in \mathbb{R}$, then*

$$\liminf_{x\to-\infty}\varphi(x) = \liminf_{x\to\infty}\varphi(x) = \inf\varphi(\mathbb{R})$$

and

$$\limsup_{x\to-\infty}\varphi(x) = \limsup_{x\to\infty}\varphi(x) = \sup\varphi(\mathbb{R}).$$

Proof. Again we restrict ourselves to the proof of the equalities with the infimum only. In view of Proposition 3.3.1, it is sufficient to show that $\liminf_{x\to-\infty}\varphi(x) = \inf\varphi(\mathbb{R})$ or $\liminf_{x\to\infty}\varphi(x) = \inf\varphi(\mathbb{R})$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of reals such that

$$\lim_{n\to\infty}\varphi(x_n) = \inf\varphi(\mathbb{R}).$$

As in the proof of Theorem 2.4.1 suppose that the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded. Then we can find its subsequence which is convergent to some $x_0 \in \mathbb{R}$. The continuity of φ implies that $\varphi(x_0) = \inf\varphi(\mathbb{R})$. By the assumption $[x_0]_\sim$ is dense and now Corollary 3.2.2 implies that φ is constant, and thus the equalities $\liminf_{x\to-\infty}\varphi(x) = \liminf_{x\to\infty}\varphi(x) = \inf\varphi(\mathbb{R})$ hold. Therefore we assume that the sequence $(x_n)_{n\in\mathbb{N}}$ is unbounded. There exists its subsequence $(y_n)_{n\in\mathbb{N}}$ such that $y_n \to -\infty$ or $y_n \to \infty$. Assume without loss of generality that the first case occurs. Then

$$\lim_{n\to\infty}\varphi(y_n) = \inf\varphi(\mathbb{R}).$$

This means that $\liminf_{x\to-\infty}\varphi(x) = \lim_{n\to\infty}\varphi(y_n)$ since $\liminf_{x\to-\infty}\varphi(x) \geq \inf\varphi(\mathbb{R})$. Hence and from Proposition 3.3.1, we have $\liminf_{x\to-\infty}\varphi(x) = \liminf_{x\to\infty}\varphi(x) = \inf\varphi(\mathbb{R})$. □

We can get immediately the following

Corollary 3.3.2. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a bounded continuous solution of equation (3.1). Assume that $\mathcal{F}$ satisfies the weak compatibility condition and $\mathcal{F}$ contains a decreasing function. If the equivalence class $[a]_\sim$ is dense in $\mathbb{R}$ for every $a \in \mathbb{R}$ and at least one of the limits $\lim_{x\to-\infty}\varphi(x)$ and $\lim_{x\to\infty}\varphi(x)$ exists, then φ is constant.*

Proof. Corollary 3.3.1 asserts that both limits $\lim_{x\to-\infty}\varphi(x)$ and $\lim_{x\to\infty}\varphi(x)$ exist. This means that $\liminf_{x\to-\infty}\varphi(x) = \limsup_{x\to-\infty}\varphi(x)$ and $\liminf_{x\to\infty}\varphi(x)$ $= \limsup_{x\to\infty}\varphi(x)$. By Proposition 3.3.2 we have

$$\inf\varphi(\mathbb{R}) = \liminf_{x\to\infty}\varphi(x) = \limsup_{x\to\infty}\varphi(x) = \sup\varphi(\mathbb{R}).$$

The above equalities imply that φ is constant. □

I would like to illustrate the results obtained in this chapter by one example which is connected with the archetypal equation.

Example 3.3.1. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a bounded continuous solution of the functional equation

$$\varphi(x) = \sum_{i\in\mathbb{Z}} p_i\varphi(-2x+i).$$

In that case we put $I = \mathbb{Z}$ and $f_i(x) = -2x + i$ for every $x \in \mathbb{R}$ and $i \in \mathbb{Z}$. At first we will check that the family $\mathcal{F}$ satisfies the weak compatibility condition. Take arbitrary $i, j \in \mathbb{Z}$ and $x \in \mathbb{R}$. Then

$$\begin{aligned}\big(f_i \circ f_j \circ f_i^{-1}\big)(x) &= -2\Big[-2\Big(-\frac{1}{2}x + \frac{i}{2}\Big) + j\Big] + i = -2(x - i + j) + i\\ &= -2x + 2i - 2j + i = -2x + 3i - 2j = f_{3i-2j}(x).\end{aligned}$$

Now we check that all the equivalence classes $[a]_\sim$, where $a \in \mathbb{R}$, are dense in $\mathbb{R}$. Fix $a \in \mathbb{R}$ and define the sequence $(D_n)_{n\in\mathbb{N}}$ of sets as follows

$$D_n = \Big\{(-1)^n\frac{a}{2^n} + \frac{s}{2^n} : \quad s \in \mathbb{Z}\Big\}$$

and

$$D = \bigcup_{n=1}^{\infty} D_n.$$

Lemma 2.3.2 implies that D is dense in $\mathbb{R}$. We are going to prove that $D \subseteq [a]_\sim$. First of all observe that $f_i^{-1}(a) = -\frac{a}{2} + \frac{i}{2} \in [a]_\sim$ for every $i \in \mathbb{Z}$, that is $D_1 \subseteq [a]_\sim$. Suppose that for some $n \in \mathbb{N}$ the set D_n is contained in $[a]_\sim$. Take any $s \in \mathbb{Z}$ and note that for every $i \in \mathbb{Z}$ we have

$$f_i^{-1}\left((-1)^n \frac{a}{2^n} + \frac{s}{2^n}\right) = (-1)^{n+1} \frac{a}{2^{n+1}} - \frac{s}{2^{n+1}} + \frac{i}{2} = (-1)^{n+1} \frac{a}{2^{n+1}} + \frac{2^n i - s}{2^{n+1}}.$$

Since i and s run through the whole $\mathbb{Z}$, the expression $2^n i - s$ can attain all integer values. Hence D_{n+1} is a subset of $[a]_\sim$. In view of the mathematical induction we get $D \subseteq [a]_\sim$, and thus $[a]_\sim$ is dense.

Observe that we can apply here Corollary 3.2.2 and Corollary 3.3.2. In summary, if the function φ attains the global extremum or there exists at least one of the limits $\lim_{x\to-\infty} \varphi(x)$ and $\lim_{x\to\infty} \varphi(x)$, then φ must be constant.

Remark 3.3.1. Note that the above equation is very close to equations considered in the previous chapter. Nevertheless we could apply the results from neither Section 2.3, nor Section 2.4 to that one. This means that the theorems obtained in the present chapter can be successfully used in the particular cases of the archetypal equation.

Chapter 4

General linear iterative equation and invariant compact sets

The current chapter of the thesis is devoted to an analyse of invariant compact sets and their influence on linear iterative equations. As it turns out these sets play a central role in a description of solutions of this kind of functional equations. In Chapter 4 we shall see that if the solution is continuous at each point of an invariant compact set, then it is constant very often. The presented part of the PhD thesis is based on the article [38].

Throughout the chapter $(\Omega, \mathcal{A}, P)$ is a probability space, (X, d) a complete metric space and $(Y, \|\cdot\|)$ a separable Banach space over the field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. We fix also a family $\{f_\omega : \ X \to X| \quad \omega \in \Omega\}$ of functions and assume that

$$\big\{\omega \in \Omega : \quad f_\omega(x) \in U\big\} \in \mathcal{A} \quad \text{for every } x \in X \text{ and open } U \subseteq X,$$

i.e. $f : \Omega \times X \to X$ defined by the formula $f(\omega, x) = f_\omega(x)$ is $\mathcal{A}$-measurable for each fixed $x \in X$. We associate with the functions $f_\omega, \omega \in \Omega$, some type of sets. We say that a set $K \subseteq X$ is *invariant* if

$$f_\omega(K) \subseteq K \quad \text{for all } \omega \in \Omega. \tag{4.1}$$

In this chapter we will be interested in bounded Borel solutions $\varphi : X \to Y$ of the functional equation

$$\varphi(x) = \int_\Omega \varphi\big(f_\omega(x)\big) P(d\omega). \tag{4.2}$$

For integrating vector functions we use the Bochner integral.

The chapter is divided into five sections. In Section 4.1 we formulate and prove our main theorem. A discussion on the existence of invariant compact sets can be found in Section 4.2. Section 4.3 contains particular cases of Theorem 4.1.1. It is worth mentioning that in this section we shall prove a generalization of the main theorem. Section 4.4 is devoted to the case when equation (4.2) is of finite order. The last part of this chapter contains examples illustrating considered topics.

4.1 The main theorem

At the beginning we remind a property of continuous functions described in the following folk theorem.

Lemma 4.1.1. *Let $(X_1, d_1), (X_2, d_2)$ be metric spaces and let $K \subseteq X_1$ be a compact set. If $f : X_1 \to X_2$ is continuous at each point of K, then for every $\varepsilon \in (0, \infty)$ there exists $\delta \in (0, \infty)$ such that for all $x \in X_1$ and $y \in K$ with $d_1(x, y) < \delta$ we have $d_2(f(x), f(y)) < \varepsilon$.*

We find out that the shape of any solution of equation (4.2) crucially depends on the behaviour of the solution on invariant compact sets. The details are described in the below result which is my main theorem in [38].

Theorem 4.1.1 (Theorem 3.1, [38])**.** *For each $\omega \in \Omega$ let $f_\omega : X \to X$ be a Matkowski type contraction with a comparison function g_ω. Assume that the functions $f_\omega, \omega \in \Omega$, or the functions $g_\omega, \omega \in \Omega$, are pairwise commuting and there exists $A \in \mathcal{A}$ such that $P(A) > 0$ and $g_{\omega_1} = g_{\omega_2}$ for all $\omega_1, \omega_2 \in A$. If $K \subseteq X$ is a nonempty compact set satisfying condition* (4.1)*, then any bounded Borel solution $\varphi : X \to Y$ of equation* (4.2)*, which is continuous at each point of K, is constant.*

Proof. Let φ be a bounded solution of equation (4.2). Then for every $n \in \mathbb{N}$ and $x \in X$ it satisfies the equality

$$\varphi(x) = \int_{\Omega^n} \varphi\big((f_{\omega_n} \circ ... \circ f_{\omega_1})(x)\big) P^n(d\omega_1, ..., d\omega_n).$$

Remember that, according to Remark 1.2.3, $(\Omega^n, \mathcal{A}^n, P^n)$ denotes the product of n copies of $(\Omega, \mathcal{A}, P)$. Fix arbitrary $\varepsilon \in (0, \infty)$. In view of Lemma 4.1.1 there exists $\delta \in (0, \infty)$ such that for every $x \in X$ and $y \in K$ with $d(x, y) < \delta$ we have $\|\varphi(x) - \varphi(y)\| < \frac{\varepsilon}{2}$. Let $M \in (0, \infty)$ be such that

$$\|\varphi(x)\| \leq M \quad \text{for every } x \in X.$$

Fix any $x_0 \in X$ and $y_0 \in K$. We are going to show that $\varphi(x_0) = \varphi(y_0)$. Remind that we have a function $g : [0, \infty) \to [0, \infty)$ which is a comparison function for all f_ω with $\omega \in A$. We can choose $k_0 \in \mathbb{N}$ fulfilling the inequality

$$g^{k_0}\big(d(x_0, y_0)\big) < \delta.$$

Take any $k \geq k_0$ and let $B_k \subseteq \Omega^k$ be the set of all sequences $(\omega_1, ..., \omega_k) \in \Omega^k$ for which the elements from A appear at most k_0 times. Then B_k can be splitted into $k_0 + 1$ pairwise disjoint sets of the form

$$C_j = \bigcup_{\sigma \in S_j} \sigma(1) \times ... \times \sigma(k),$$

where $S_j := \big\{\sigma : \{1, 2, ..., k\} \to \{A, \Omega \setminus A\} | \quad \text{card}\, \sigma^{-1}(\{A\}) = j\big\}$ for each $j = \ 0, 1, ..., k_0$. In other words C_j contains all sequences $(\omega_1, ..., \omega_k) \in \Omega^k$ for which members of A appear exactly j times for all $j = 0, 1, ..., k_0$. Then we can write B_k down as

$$B_k = \bigcup_{j=0}^{k_0} C_j.$$

Hence $B_k \in \mathcal{A}^k$ and furthermore

$$P^k(B_k) = \sum_{j=0}^{k_0} \binom{k}{j} P(A)^j \big(1 - P(A)\big)^{k-j}.$$

Now we are going to prove that $\lim_{k\to\infty} P^k(B_k) = 0$. At first note that for sufficiently large k we have

$$\binom{k}{0} < \binom{k}{1} < ... < \binom{k}{k_0}.$$

Hence and from the fact that $P(A) \in (0, 1)$ we get

$$\begin{aligned}
\lim_{k\to\infty} P^k(B_k) &\leq \lim_{k\to\infty} \sum_{j=0}^{k_0} \binom{k}{k_0} P(A)^j \big(1 - P(A)\big)^{k-j} \\
&\leq \lim_{k\to\infty} \sum_{j=0}^{k_0} \binom{k}{k_0} \max\big\{P(A), 1 - P(A)\big\}^k
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{k\to\infty} (k_0+1)\binom{k}{k_0} \max\big\{P(A), 1-P(A)\big\}^k \\
&= \lim_{k\to\infty} (k_0+1)\frac{k(k-1)...(k-(k_0-1))}{k_0!} \max\big\{P(A), 1-P(A)\big\}^k \\
&=\frac{k_0+1}{k_0!}\lim_{k\to\infty} k(k-1)...(k-(k_0-1)) \max\big\{P(A), 1-P(A)\big\}^k \\
&\leq\frac{k_0+1}{k_0!}\lim_{k\to\infty} k^{k_0} \max\big\{P(A), 1-P(A)\big\}^k = 0,
\end{aligned}
$$

and thus $\lim_{k\to\infty} P^k(B_k) = 0$. Therefore we can choose k in such a way that

$$
P^k(B_k) < \frac{\varepsilon}{4M}.
$$

Note that for all $(\omega_1, ..., \omega_k) \in \Omega^k$ we have $(f_{\omega_k} \circ ... \circ f_{\omega_1})(y_0) \in K$ because of (4.1). Then we have

$$
d\big((f_{\omega_k} \circ ... \circ f_{\omega_1})(x_0), (f_{\omega_k} \circ ... \circ f_{\omega_1})(y_0)\big) \leq (g_{\omega_k} \circ ... \circ g_{\omega_1})\big(d(x_0, y_0)\big). \quad (4.3)
$$

Assume that the functions $g_\omega, \omega \in \Omega$, are pairwise commuting and take the sequence $(\omega_1, ..., \omega_k) \in \Omega^k \setminus B_k$. Without loss of generality we may assume that $\omega_1, ..., \omega_{k_0} \in A$. Then, in view of the definition of k_0, we have

$$
(g_{\omega_k}\circ...\circ g_{\omega_1})\big(d(x_0, y_0)\big) \leq (g_{\omega_{k_0}}\circ...\circ g_{\omega_1})\big(d(x_0, y_0)\big) = g^{k_0}\big(d(x_0, y_0)\big) < \delta. \quad (4.4)
$$

Now consider the case when the functions $f_\omega, \omega \in \Omega$, are pairwise commuting and fix aribtrarily $(\omega_1, ..., \omega_k) \in \Omega^k \setminus B_k$. We can change the order of the functions $f_{\omega_k}, ..., f_{\omega_1}$ in inequality (4.3) in such a way that the first k_0 indices come from A. Then we obtain (4.4) again.

Note that (4.3) and (4.4) imply

$$
d\big((f_{\omega_k} \circ ... \circ f_{\omega_1})(x_0), (f_{\omega_k} \circ ... \circ f_{\omega_1})(y_0)\big) < \delta \quad \text{for all } (\omega_1, ..., \omega_k) \in \Omega^k \setminus B_k
$$

Therefore, we have

$$
\begin{aligned}
&\|\varphi(x_0) - \varphi(y_0)\| \\
&\leq \int_{\Omega^k} \Big\| \varphi\big((f_{\omega_k} \circ ... \circ f_{\omega_1})(x_0)\big) - \varphi\big((f_{\omega_k} \circ ... \circ f_{\omega_1})(y_0)\big) \Big\| P^k(d\omega_1, ..., d\omega_k)
\end{aligned}
$$

$$= \int_{\Omega^k \setminus B_k} \left\| \varphi\big((f_{\omega_k} \circ ... \circ f_{\omega_1})(x_0)\big) - \varphi\big((f_{\omega_k} \circ ... \circ f_{\omega_1})(y_0)\big) \right\| P^k(d\omega_1, ..., d\omega_k)$$
$$+ \int_{B_k} \left\| \varphi\big((f_{\omega_k} \circ ... \circ f_{\omega_1})(x_0)\big) - \varphi\big((f_{\omega_k} \circ ... \circ f_{\omega_1})(y_0)\big) \right\| P^k(d\omega_1, ..., d\omega_k)$$
$$\leq \frac{1}{2}\varepsilon P^k(\Omega^k \setminus B_k) + 2MP^k(B_k) < \frac{\varepsilon}{2} + 2M\frac{\varepsilon}{4M} = \varepsilon.$$

This means that φ is constant and the proof is complete. □

Remark 4.1.1. Note that in Theorem 4.1.1 we may assume less, namely that almost all the functions $f_\omega, \omega \in \Omega$, are Matkowski type contractions. And similarly:

$$f_\omega(K) \subseteq K \quad \text{for almost all } \omega \in \Omega$$

and almost all the functions $f_\omega, \omega \in \Omega$, $[g_\omega, \omega \in \Omega,]$ are pairwise commuting. The proof remains the same, we only need to consider some full measure subset instead of the whole Ω.

We see that some assumptions here are technical and they can be sometimes difficult in checking. Therefore we will present in Section 4.3 a lot of their more useful particular cases.

Continuity at each point of the set K is essential in Theorem 4.1.1. If we omit this assumption, then we can get non-constant solutions of equation (4.2).

Example 4.1.1 (Example 2, [38])**.** Let $\mathbb{R}$ be equipped with the natural metric and consider the functional equation

$$\varphi(x) = \frac{\varphi(\frac{x}{2}) + \varphi(\frac{x}{3})}{2},$$

where $\varphi : \mathbb{R} \to \mathbb{R}$. It is a particular case of equation (4.2) and the assumptions of Theorem 4.1.1, besides the continuity of the solution at each point of K, are satisfied. We note that $K = \{0\}$ is an invariant compact set with respect to the functions $\mathbb{R} \ni x \to x/2$ and $\mathbb{R} \ni x \to x/3$. One can check that the function $\varphi : \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi(x) = \begin{cases} 1, & \text{if } x \in \mathbb{R} \setminus \{0\}, \\ 0, & \text{if } x = 0, \end{cases}$$

is a solution of the above equation. The function φ has only one discontinuity point, namely 0.

4.2 Existence of invariant compact sets

We have seen that invariant compact sets are very useful in the theory of linear functional equations. Therefore it is important to know how to find them. The theorem below may be treated as the starting point of the research in this direction.

We are going to examine the existence of invariant compact sets. The next simple lemma deals with the case when the functions $f_\omega, \omega \in \Omega$, are pairwise commuting.

Lemma 4.2.1. *For each $\omega \in \Omega$ let $f_\omega : X \to X$ be a Matkowski type contraction. Assume that the functions $f_\omega, \omega \in \Omega$, are pairwise commuting. Then all of them have a common fixed point.*

Proof. In view of Theorem 1.1.1 each Matkowski type contraction has a unique fixed point. Take any $\omega, \bar{\omega} \in \Omega$ and let $\xi \in X$ be the fixed point of f_ω. We will show that ξ is a fixed point of $f_{\bar{\omega}}$, too. Note that

$$f_{\bar{\omega}}(\xi) = f_{\bar{\omega}}(f_\omega(\xi)) = f_\omega(f_{\bar{\omega}}(\xi)).$$

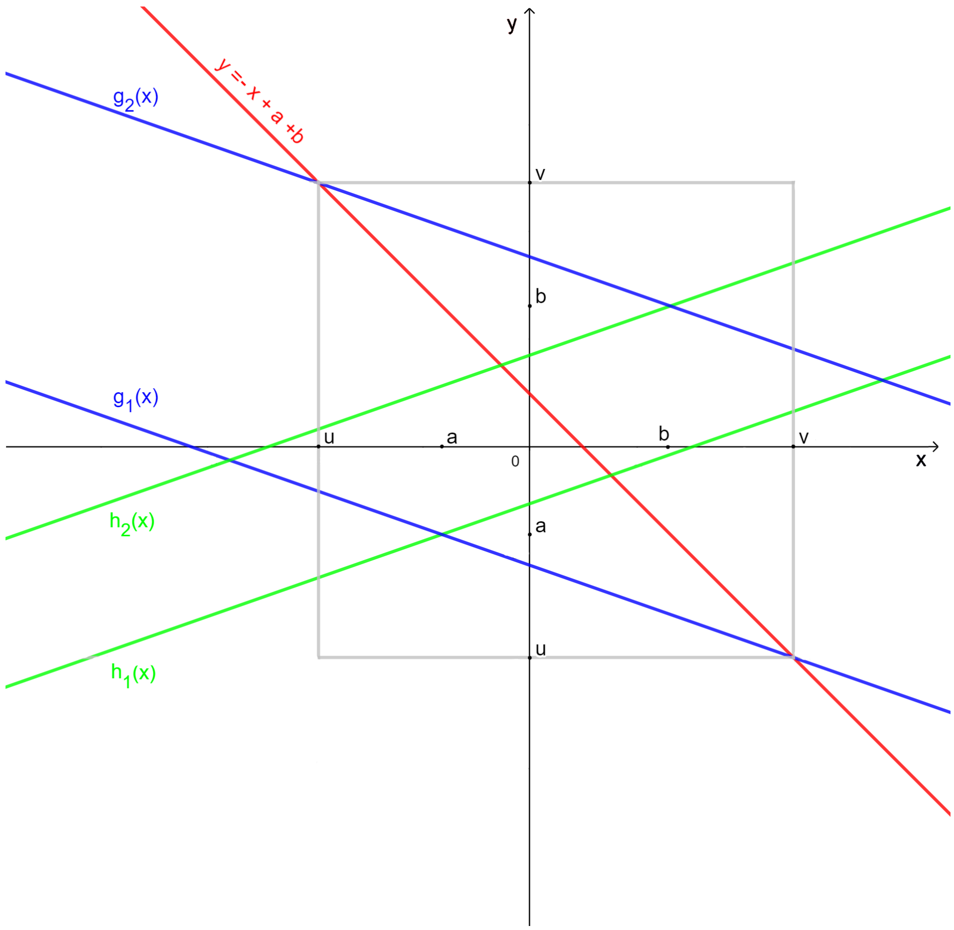
The above equality means that $f_{\bar{\omega}}(\xi)$ is a fixed point of the function f_ω. Since f_ω has exactly one fixed point, we see that $f_{\bar{\omega}}(\xi) = \xi$, i.e. ξ is a fixed point of $f_{\bar{\omega}}$. □

The above folk lemma implies that the problem of the existence of invariant compact sets becomes trivial when the functions $f_\omega, \omega \in \Omega$, are pairwise commuting.

The next theorem describes how to get the invariant compact sets in the case when X is the real line equipped with the natural metric. This is the second main result of [38].

Theorem 4.2.1 (Theorem 3.3, [38]). *For each $\omega \in \Omega$ let $f_\omega : \mathbb{R} \to \mathbb{R}$ satisfy the Lipschitz condition with the same constant $L \in (0, 1)$. If there is a bounded set containing fixed points of the functions $f_\omega, \omega \in \Omega$, then there exists a nonempty compact subset of $\mathbb{R}$ fulfilling condition* (4.1).

Proof. Note that if the functions $f_\omega, \omega \in \Omega$, have a common fixed point $\xi \in \mathbb{R}$, then $K = \{\xi\}$ is a compact set satisfying condition (4.1). Thus we exclude this case in the rest of the proof.



Assume that all fixed points of the functions $f_\omega, \omega \in \Omega$, are contained in an interval $[a, b]$, where $-\infty < a < b < \infty$. We define functions $g_1, g_2 : \mathbb{R} \to \mathbb{R}$ by the equalities

$$g_1(x) = -L(x-a) + a \quad \text{and} \quad g_2(x) = -L(x-b) + b.$$

The solutions of the equations

$$g_1(x) = -x + a + b \quad \text{and} \quad g_2(x) = -x + a + b$$

are equal $v = \frac{b-La}{1-L}$ and $u = \frac{a-Lb}{1-L}$, respectively. Furthermore we define also functions $h_1, h_2 : \mathbb{R} \to \mathbb{R}$ by the formulas

$$h_1(x) = L(x-a) + a \quad \text{and} \quad h_2(x) = L(x-b) + b.$$

We are going to prove that $g_i(x), h_i(x) \in [u, v]$ if $x \in [u, v]$ and $i \in \{1, 2\}$. At the beginning note that $h_1(u) > u$ and $h_2(v) < v$. Observe that h_1 and h_2

are increasing and $h_1(x) < h_2(x)$ for all $x \in \mathbb{R}$. Therefore, if we take $x \in [u, v]$, then we will get

$$u < h_1(u) \leq h_1(x) < h_2(x) \leq h_2(v) < v.$$

In other words, we have proven that $h_i([u, v]) \subseteq [u, v]$ for $i \in \{1, 2\}$. We show that the analogical inclusions hold for g_1 and g_2. Observe that

$$g_2(u) = v \quad \text{and} \quad g_1(v) = u.$$

Since g_1 and g_2 are decreasing and the inequality $g_1(x) < g_2(x)$ is satisfied on the whole real line, we have

$$u = g_1(v) \leq g_1(x) < g_2(x) \leq g_2(u) = v \quad \text{for every } x \in [u, v].$$

Now we are going to prove that the interval $[u, v]$ is a desired invariant set. Let $\omega \in \Omega$ be fixed and take a fixed point $\xi \in [a, b]$ of f_ω. Let $x \in [u, v]$. We shall show that $f_\omega(x) \in [u, v]$. Reasoning is divided into two complementary cases.

Case 1: Assume that $x \in [\xi, v]$. Since f_ω satisfies the Lipschitz condition with constant L, then we have

$$f_\omega(x) \leq L|x - \xi| + \xi = L(x - \xi) + \xi \leq L(x - b) + b = h_2(x) \leq h_2(v) < v$$

and

$$f_\omega(x) \geq -L|x - \xi| + \xi = -L(x - \xi) + \xi \geq -L(x - a) + a = g_1(x) \geq u.$$

Case 2: Assume that $x \in [u, \xi)$. Then

$$f_\omega(x) \leq L|x - \xi| + \xi = -L(x - \xi) + \xi \leq -L(x - b) + b = g_2(x) \leq v$$

and

$$f_\omega(x) \geq -L|x - \xi| + \xi = L(x - \xi) + \xi \geq L(x - a) + a = h_1(x) \geq h_1(u) > u.$$

In conclusion, we have $f_\omega([u, v]) \subseteq [u, v]$ and the proof is complete. □

4.3 Particular cases of the main theorem

As we said the assumptions of Theorem 4.1.1 can be sometimes difficult in checking. Therefore we are going to present the particular cases of this theorem for which a part of the assumptions are automatically satisfied. This section concerns mainly the existence of sets A and K with desired properties. Lemma 4.2.1 guarantees the existence of the invariant compact set when the functions $f_\omega, \omega \in \Omega$, are pairwise commuting. Thus we confine ourselves to the case when the functions $g_\omega, \omega \in \Omega$, are pairwise commuting; there are only two exceptions, namely Corollary 4.3.4 and Corollary 4.3.8. The presented corollaries come from [38].

If we restrict our analysis to the case when $(X, d) = (\mathbb{R}, |\cdot|)$, where $|\cdot|$ denotes the Euclidean norm, then we can take any bounded interval instead of a compact K. The closure of any subset E of $\mathbb{R}$ will be denoted by $\mathrm{cl}E$. We have

Corollary 4.3.1 (Corollary 4.1, [38]). *For each $\omega \in \Omega$ let $f_\omega : \mathbb{R} \to \mathbb{R}$ be a Matkowski type contraction with a comparison function g_ω. Assume that the functions $g_\omega, \omega \in \Omega$, are pairwise commuting and there exists $A \in \mathcal{A}$ such that $P(A) > 0$ and $g_{\omega_1} = g_{\omega_2}$ for all $\omega_1, \omega_2 \in A$. If $I \subseteq \mathbb{R}$ is a nonempty bounded interval such that*

$$f_\omega(I) \subseteq I \quad \textit{for all } \omega \in \Omega,$$

then any bounded Borel solution $\varphi : X \to Y$ of equation (4.2)*, which is continuous at each point of* $\mathrm{cl}I$*, is constant.*

Proof. The set $\mathrm{cl}I$ is compact. If $\omega \in \Omega$ and $f_\omega(I) \subseteq I$, then the continuity of f_ω implies $f_\omega(\mathrm{cl}I) \subseteq \mathrm{cl}f_\omega(I) \subseteq \mathrm{cl}I$. Now it is enough to apply Theorem 4.1.1. □

If the whole space X is compact, then $K := X$ clearly satisfies (4.1), and thus we get immediately the following

Corollary 4.3.2 (Corollary 4.2, [38]). *Assume that X is compact. For each $\omega \in \Omega$ let $f_\omega : X \to X$ be a Matkowski type contraction with a comparison function g_ω. Assume that the functions $g_\omega, \omega \in \Omega$, are pairwise commuting and there exists $A \in \mathcal{A}$ such that $P(A) > 0$ and $g_{\omega_1} = g_{\omega_2}$ for all $\omega_1, \omega_2 \in \Omega$. Then any bounded continuous solution $\varphi : X \to Y$ of equation* (4.2) *is constant.*

There is another particular case of Theorem 4.1.1 when we can find an invariant compact set in a trivial way, namely if the functions $f_\omega, \omega \in \Omega$, have a common fixed point $\xi \in X$. In the case like this condition (4.1) is satisfied since we can use $K = \{\xi\}$. In fact, we have the following

Corollary 4.3.3 (Corollary 4.3, [38]). *For each $\omega \in \Omega$ let $f_\omega : X \to X$ be a Matkowski type contraction with a comparison function g_ω. Assume that the functions $g_\omega, \omega \in \Omega$, are pairwise commuting and there exists $A \in \mathcal{A}$ such that $P(A) > 0$ and $g_{\omega_1} = g_{\omega_2}$ for all $\omega_1, \omega_2 \in A$. If there exists $\xi \in X$ such that*

$$f_\omega(\xi) = \xi \quad \textit{for all } \omega \in \Omega,$$

then any bounded Borel solution $\varphi : X \to Y$ of equation (4.2)*, which is continuous at ξ, is constant.*

As we have seen in Lemma 4.2.1, Matkowski type contractions have a common fixed point if they are pairwise commuting. In such a case we have

Corollary 4.3.4 (Corollary 4.4, [38]). *For each $\omega \in \Omega$ let $f_\omega : X \to X$ be a Matkowski type contraction with a comparison function g_ω. Assume that the functions $f_\omega, \omega \in \Omega$, are pairwise commuting and there exists $A \in \mathcal{A}$ such that $P(A) > 0$ and $g_{\omega_1} = g_{\omega_2}$ for all $\omega_1, \omega_2 \in A$. Then any bounded Borel solution $\varphi : X \to Y$ of equation* (4.2)*, which is continuous at the common fixed point of the functions $f_\omega, \omega \in \Omega$, is constant.*

We can consider also the case when the inner functions $f_\omega, \omega \in \Omega$, are Matkowski type contractions with the same comparison function g. If this case occurs, then Theorem 4.1.1 takes the following simple form.

Corollary 4.3.5 (Corollary 4.5, [38]). *For each $\omega \in \Omega$ let $f_\omega : X \to X$ be a Matkowski type contraction with a comparison function g. If K is a nonempty compact subset of X satisfying condition* (4.1)*, then any bounded Borel solution $\varphi : X \to Y$ of equation* (4.2)*, which is continuous at each point of K, is constant.*

If the inner functions appearing in equation (4.2) are classical contractions, then linear functions can be chosen as comparison functions, and thus they are pairwise commuting.

Corollary 4.3.6 (Corollary 4.6, [38]). *For each $\omega \in \Omega$ let $f_\omega : X \to X$ be a contraction with Lipschitz constant $c_\omega \in (0,1)$. Assume that there exists $A \in \mathcal{A}$ such that $P(A) > 0$ and $c_{\omega_1} = c_{\omega_2}$ for all $\omega_1, \omega_2 \in A$. If $K \subseteq X$ is a nonempty compact set satisfying condition* (4.1), *then any bounded Borel solution $\varphi : X \to Y$ of equation* (4.2), *which is continuous at each point of K, is constant.*

If we go back on the real line and assume that the inner functions $f_\omega, \omega \in \Omega$, are contractions with the same Lipschitz constant L, then, in view of Theorem 4.2.1, we can sometimes express the invariant compact set by the explicit formula.

Corollary 4.3.7 (Corollary 4.7, [38]). *For each $\omega \in \Omega$ let $f_\omega : \mathbb{R} \to \mathbb{R}$ satisfy the Lispchitz condition with constant $L \in (0,1)$. If there exists an interval $[a,b]$, where $-\infty < a < b < \infty$, containing fixed points of the functions $f_\omega, \omega \in \Omega$, then any bounded Borel solution $\varphi : \mathbb{R} \to \mathbb{R}$ of equation* (4.2), *which is continuous at each point of the interval $\left[\frac{a-Lb}{1-L}, \frac{b-La}{1-L}\right]$, is constant.*

In contrast to the above corollaries, the next one is a generalization of Theorem 4.1.1. During a discussion on the Seminar on Functional Equations at the University of Zielona Góra, Janusz Matkowski noticed that the argument presented in the proof of Theorem 4.1.1 could be applied to a more general equation than (4.2). We have the following extension of the main result.

Corollary 4.3.8 (Corollary 4.8, [38]). *Let $\delta : \Omega \to \mathbb{R}$ be a non-negative $\mathcal{A}$-measurable function such that*

$$\int_\Omega \delta(\omega) P(d\omega) = 1.$$

For each $\omega \in \Omega$ let $f_\omega : X \to X$ be a Matkowski type contraction with a comparison function g_ω. Assume that the functions $f_\omega, \omega \in \Omega$, or the functions $g_\omega, \omega \in \Omega$, are pairwise commuting and there exists $A \in \mathcal{A}$ such that $\int_A \delta dP > 0$ and $g_{\omega_1} = g_{\omega_2}$ for all $\omega_1, \omega_2 \in A$. If $K \subseteq X$ is a nonempty compact set satisfying condition (4.1), *then any bounded Borel solution $\varphi : X \to Y$ of the equation*

$$\varphi(x) = \int_\Omega \delta(\omega) \varphi\big(f_\omega(x)\big) P(d\omega), \tag{4.5}$$

which is continuous at each point of K, is constant.

Proof. Define a probability measure Q on Ω by the formula

$$Q(A) = \int_A \delta(\omega) P(d\omega).$$

Then the measure Q is absolutely continuous with respect to P. Applying Lemma 1.2.1 we get

$$\int_\Omega \delta(\omega)\varphi\big(f_\omega(x)\big)P(d\omega) = \int_\Omega \varphi\big(f_\omega(x)\big)Q(d\omega) \quad \text{for every } x \in X.$$

Hence we can rewrite equation (4.5) in the form

$$\varphi(x) = \int_\Omega \varphi\big(f_\omega(x)\big)Q(d\omega).$$

Since all the assumptions of Theorem 4.1.1 are satisfied, we can apply it, taking Q instead of P, to the above functional equation and the proof of this corollary is finished. □

We pass to the case when the integral in equation (4.2) reduces to a series. This phenomena occurs when the measure P is discrete with the infinite support (a disscusion on equations of finite order is postponed to the next section).

Corollary 4.3.9 (Corollary 4.9, [38]). *Let $(p_n)_{n\in\mathbb{N}}$ be a sequence of positive reals summing up to one. For each $n \in \mathbb{N}$ let $f_n : X \to X$ be a Matkowski type contraction with a comparison function g_n. Assume that the functions $g_n, n \in \mathbb{N}$, are pairwise commuting. If $K \subseteq X$ is a nonempty compact set such that*

$$f_n(K) \subseteq K \quad \textit{for all } n \in \mathbb{N},$$

then any bounded Borel solution $\varphi : X \to Y$ of the equation

$$\varphi(x) = \sum_{n=1}^{\infty} p_n \varphi\big(f_n(x)\big),$$

which is continuous at each point of K, is constant.

Proof. Put $\Omega = \mathbb{N}$, $\mathcal{A} = 2^{\mathbb{N}}$ and define $P(\{n\}) = p_n$ for $n \in \mathbb{N}$. We can apply Theorem 4.1.1 taking any singleton in $\mathbb{N}$ as the set A. □

4.4 Equation of finite order

Now we consider a very particular case of equation (4.2) which has a finite number of terms on the right-hand side. In this part we fix $N \in \mathbb{N}$ and $p_1, ..., p_N \in (0,1)$ such that $p_1 + ... + p_N = 1$. We shall be interested in solutions $\varphi : X \to Y$ of the equation

$$\varphi(x) = \sum_{i=1}^{N} p_i \varphi\big(f_i(x)\big), \tag{4.6}$$

where $f_1, ..., f_N : X \to X$ are given maps. We have seen in Corollary 4.3.5 that if we can choose a common comparison function for the functions $f_\omega, \omega \in \Omega$, then the situation is much simpler. For the equation of finite order such a choice is always possible. The below lemma shows how to make it.

Lemma 4.4.1. *Assume that $g_1, ..., g_N : [0, \infty) \to [0, \infty)$ are the comparison functions. Then $g := \max\{g_1, ..., g_N\}$ is also a comparison function.*

Proof. The facts that: g is non-decreasing, $g(0) = 0$ and $g(t) < t$ for all positive t are trivial and we omit their proofs. We show that for every $t \in [0, \infty)$ the sequence $(g^n(t))_{n \in \mathbb{N}}$ converges to 0.

Fix arbitrarily $i \in \{1, 2, ..., N\}$ and $u \in (0, \infty)$. In the first step we show that if $\lim_{t \to u^+} g_i(t) = u$, then there exists $\varepsilon \in (0, \infty)$ such that the function g_i is constant on the interval $(u, u + \varepsilon)$. Assume on the contrary that the equality $\lim_{t \to u^+} g_i(t) = u$ holds and there is no $\varepsilon \in (0, \infty)$ for which $g_{i|(u,u+\varepsilon)}$ is constant. Take any $v > u$. Note that if $t \in (u, v)$, then $g_i(t) \in (u, v)$, too. Hence the sequence $(g_i^n(t))_{n \in \mathbb{N}}$ cannot be convergent to 0 if t comes from the interval (u, v). This contradicts the assumption saying that g_i is a comparison function. Therefore we have proven that if $\lim_{t \to u^+} g_i(t) = u$, then we can always find a positive ε such that $g_{i|(u,u+\varepsilon)}$ is constant. Furthermore we can notice that $g_i(t) = u$ for all $t \in (u, u + \varepsilon)$.

Fix any $t_0 \in (0, \infty)$. We are going to prove that the sequence $(g^n(t_0))_{n \in \mathbb{N}}$ converges to 0. Since $g(t_0) < t_0$, the sequence $(g^n(t_0))_{n \in \mathbb{N}}$ is non-increasing. Moreover, we know that g is bounded below by 0. Hence the sequence $(g^n(t_0))_{n \in \mathbb{N}}$ is convergent. We denote its limit by u and suppose that $u > 0$. From the monotonicity of g we have

$$g(u) \leq \lim_{t \to u^+} g(t). \tag{4.7}$$

Further, note that

$$\lim_{t \to u^+} g(t) = \lim_{n \to \infty} g(g^n(t_0)) = \lim_{n \to \infty} g^{n+1}(t_0) = u. \tag{4.8}$$

First of all assume that inequality (4.7) is strict. Equality (4.8) and the definition of g imply that $\lim_{t \to u^+} g_i(t) = u$ for some $i \in \{1, 2, ..., N\}$. As we proved before, there exists $\varepsilon_1 \in (0, \infty)$ such that the function $g_{i|(u,u+\varepsilon_1)}$ is constant and equal to u. Furthermore we see that

$$\lim_{t \to u^+} g_j(t) \leq \lim_{t \to u^+} g_i(t) = u \quad \text{for every } j \in \{1, 2, ..., N\}.$$

Since $g_1, g_2, ..., g_N$ are comparison functions and the above inequalities holds, we can find $\varepsilon \in (0, \varepsilon_1)$ such that

$$g_j(t) \leq u \quad \text{for all } t \in (u, u + \varepsilon) \text{ and } j \in \{1, 2, ..., N\}.$$

This means that

$$g(t) = u \quad \text{for every } t \in (u, u + \varepsilon).$$

Let $n_0 \in \mathbb{N}$ be such that $g^{n_0}(t_0) \in (u, u + \varepsilon)$. Then $g^{n_0+1}(t_0) = u$ and

$$g^{n_0+2}(t_0) = g(u) < u$$

since we have strict inequality in (4.7). It is a contradiction with the condition saying that

$$g^n(t_0) \geq u \quad \text{for every } n \in \mathbb{N}.$$

Hence inequality (4.7) cannot be strict and we have $g(u) = u$. But it contradicts the inequality

$$g(t) < t \quad \text{for every } t \in (0, \infty).$$

Therefore $u = 0$ and the proof has been completed. □

The fact described in Lemma 4.4.1 is well known in the theory of GIFS but I did not find its proof in the literature, and thus the above reasoning is presented for the sake of completeness.

Remark 4.4.1. The consequences of this lemma are very useful. If we knew that $f_1, ..., f_N$ are Matkowski type contractions with the comparison functions $g_1, ..., g_N$, respectively, then we could take the function g defined as in Lemma 4.4.1, and then the mappings from the system $\{f_1, ..., f_N\}$ would be Matkowski type contractions with the same comparison function g. Moreover, note that condition (4.1) reduces to

$$f_i(K) \subseteq K \quad \text{for all } i = 1, 2, ..., N$$

in this case. We know from Theorem 1.1.2 that for such finite systems of functions there always exists a compact subset of X, called *attractor*, fulfilling condition (1.1) clearly stronger than (4.1).

We are in a position to prove the last corollary.

Corollary 4.4.1 (Corollary 5.4, [38]). *For each $i \in \{1, 2, ..., N\}$ let $f_i : X \to X$ be a Matkowski type contraction with a comparison function g_i. Then any bounded Borel solution $\varphi : X \to Y$ of equation* (4.6)*, which is continuous at each point of the attractor of the system $\{f_1, ..., f_N\}$, is constant.*

Proof. Let $g := \max\{g_1, ..., g_N\}$. In view of Lemma 4.4.1 the function g is a comparison function. We see that the mappings $f_1, ..., f_N$ are Matkowski type contractions with g. Theorem 1.1.2 implies that there exists a compact set $K \subseteq X$ such that

$$\bigcup_{i=1}^{N} f_i(K) = K.$$

Hence $f_i(K) \subseteq K$ for every $i = 1, 2, ..., N$. The thesis is a simple consequence of Corollary 4.3.5. □

4.5 Examples

In the last part of Chapter 4, I would like to present some interesting examples. The below equations were considered by me when I was analysing an influence of invariant compact sets on continuous solutions of linear functional equations. I also examined the cases which are not covered by Theorem 4.1.1. This part of my work have never been published before and these examples are not included in the manuscript [38], too.

The first example is an illustration of applications of Theorem 4.1.1. Furthermore we have a possibility to compare Theorems 2.2.4 and 4.1.1.

Example 4.5.1. Let $\beta : \Omega \to \mathbb{R}$ be a random variable with a uniform distribution on the interval $\left[0, \frac{1}{2}\right]$. Suppose, for this example, that X and Y are the set of real numbers endowed with Euclidean metric and Euclidean norm, respectively. We define the inner functions by the equality

$$f_\omega(x) := \frac{1}{2}x + \beta(\omega) \quad \text{for every } \omega \in \Omega \text{ and } x \in \mathbb{R}.$$

Observe that the functions $f_\omega, \omega \in \Omega$, are contractions with the Lipschitz constant equal to 1/2. Take any bounded Borel solution $\varphi : \mathbb{R} \to \mathbb{R}$ of the equation

$$\varphi(x) = \int_\Omega \varphi\left(\frac{1}{2}x + \beta(\omega)\right) P(d\omega).$$

We check that the assumptions of Theorem 2.2.4 are satisfied for such an equation. At the beginning note that the hypothesis (H) holds. The parameter K is equal to

$$K = \int_0^{\frac{1}{2}} 2\ln\left(\frac{1}{2}\right) dt = -\ln 2 < 0.$$

Moreover the integral $\iint_{\mathbb{R}^2} \ln\big(\max\{|b|, 1\}\big)\mu(da, db)$ reduces in that case to

$$\int_0^{\frac{1}{2}} 2\ln\big(\max\{t, 1\}\big) dt = \ln 1 = 0 < \infty.$$

Therefore Theorem 2.2.4 (i) implies that φ is constant if φ is continuous. We will see that it is enough to impose on φ a weaker regularity condition. Observe that the interval $[0, 1]$ is an invariant compact set. We have

$$f_\omega\big([0,1]\big) = \frac{1}{2}[0,1] + \beta(\omega) = \left[0, \frac{1}{2}\right] + \beta(\omega) \subseteq [0,1] \quad \text{for every } \omega \in \Omega,$$

since values of the random variable β lie in the interval $\big[0, 1/2\big]$. Now Corollary 4.3.6 asserts that if φ is continuous at each point of $[0, 1]$, then it is constant.

Remark 4.5.1. In this case Corollary 4.3.6 is more suitable than Theorem 2.2.4 – we assumed that φ is continuous only at each point of a compact set. In general the weakness of Corollary 4.3.6 is that we cannot use it if the functions $f_\omega, \omega \in \Omega$, are not contractions on the set of positive measure.

In contrast to Example 4.5.1 we cannot apply Theorem 4.1.1 below. The functions $f_\omega, \omega \in \Omega$, in equation (4.9) have two fixed points, and thus they cannot be Matkowski type contractions. Nevertheless we will see that invariant compact sets exist and again they play important role when we consider solutions which are continuous at each point of the invariant compact sets.

Proposition 4.5.1. *Let $\varphi : [0,\infty) \to \mathbb{R}$ be a solution of the equation*

$$\varphi(x) = \frac{\varphi(x^2) + \varphi(\sqrt{x})}{2}. \tag{4.9}$$

If φ is continuous at 0 and 1, then φ is constant.

Proof. First of all note that since φ satisfies equation (4.9), then for every $x \in [0,\infty)$ we have

$$\begin{aligned}\varphi(x) =& \frac{1}{2}\varphi(x^2) + \frac{1}{2}\varphi(\sqrt{x}) = \frac{1}{2}\left[\frac{\varphi(x^4) + \varphi(x)}{2}\right] + \frac{1}{2}\left[\frac{\varphi(x) + \varphi(\sqrt[4]{x})}{2}\right] \\ =& \frac{1}{4}\varphi(x^4) + \frac{1}{4}\varphi(\sqrt[4]{x}) + \frac{1}{2}\varphi(x).\end{aligned}$$

Hence

$$\varphi(x) = \frac{\varphi(x^4) + \varphi(\sqrt[4]{x})}{2} \quad \text{for every } x \in [0,\infty).$$

One can prove inductively that for every $n \in \mathbb{N}$ the function φ satisfies the equality

$$\varphi(x) = \frac{\varphi(x^{2^n}) + \varphi(\sqrt[2^n]{x})}{2} \quad \text{for every } x \in [0,\infty). \tag{4.10}$$

Take any $x \in (0,1)$. We have $\lim_{n\to\infty} x^{2^n} = 0$ and $\lim_{n\to\infty} \sqrt[2^n]{x} = 1$. Since φ is continuous at 0 and 1, then tending with n to ∞ in equality (4.10) we get

$$\varphi(x) = \frac{\varphi(0) + \varphi(1)}{2}.$$

In other words, $\varphi_{|(0,1)}$ is constant. This equality, jointly with the continuity of φ at the endpoints of the interval $(0,1)$, gives $\varphi(0) = \varphi(1)$. Therefore $\varphi_{|[0,1]}$ is constant.

We ae going to show that $\varphi_{|(1,\infty)}$ is constant. Let $\varepsilon \in (0,\infty)$ be taken arbitrary. Since φ is continuous at 1, we can find $\delta \in (0,\infty)$ such that

$$|\varphi(t) - \varphi(1)| < \frac{\varepsilon}{6} \quad \text{for every } t \in (1, 1+\delta).$$

Then for all $s,t \in (1,1+\delta)$ we have the following estimation

$$\begin{aligned}|\varphi(s) - \varphi(t)| &= |\varphi(s) - \varphi(1) + \varphi(1) - \varphi(t)| \\ &\le |\varphi(s) - \varphi(1)| + |\varphi(t) - \varphi(1)| < \frac{\varepsilon}{3}.\end{aligned} \tag{4.11}$$

Fix $s, t \in (1, \infty)$. Applying equality (4.10) to s and t we get

$$\varphi(s^{2^n}) = 2\varphi(s) - \varphi(\sqrt[2^n]{s}) \quad \text{and} \quad \varphi(t^{2^n}) = 2\varphi(t) - \varphi(\sqrt[2^n]{t}) \quad \text{for every } n \in \mathbb{N}.$$

Therefore

$$\left|\varphi(s^{2^n}) - \varphi(t^{2^n})\right| \leq 2\left|\varphi(s) - \varphi(t)\right| + \left|\varphi(\sqrt[2^n]{s}) - \varphi(\sqrt[2^n]{t})\right| \quad \text{for all } n \in \mathbb{N}.$$

Fix arbitrary $n \in \mathbb{N}$. Take $s_0, t_0 \in (1, 1+\delta)$ and $k \in \mathbb{N}$ in such a way that $s_0^{2^k} = s$ and $t_0^{2^k} = t$. Observe that $\sqrt[2^n]{s_0}, \sqrt[2^n]{t_0} \in (1, 1+\delta)$, too. Moreover, in view of (4.11) we have $|\varphi(s_0) - \varphi(t_0)| < \frac{\varepsilon}{3}$ and $|\varphi(\sqrt[2^{n+k}]{s_0}) - \varphi(\sqrt[2^{n+k}]{t_0})| < \frac{\varepsilon}{3}$. Hence

$$\begin{aligned}\left|\varphi(s^{2^n}) - \varphi(t^{2^n})\right| =& \left|\varphi(s_0^{2^{n+k}}) - \varphi(t_0^{2^{n+k}})\right| \\ \leq& 2\left|\varphi(s_0) - \varphi(t_0)\right| + \left|\varphi(\sqrt[2^{n+k}]{s_0}) - \varphi(\sqrt[2^{n+k}]{t_0})\right| < \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

The above inequality implies $\varphi(s^{2^n}) = \varphi(t^{2^n})$. Since n and the arguments s, t were fixed arbitrarily we have proven that

$$\varphi(s^{2^n}) = \varphi(t^{2^n}) \quad \text{for every } n \in \mathbb{N} \text{ and } s, t \in (1, \infty).$$

Let $x, y \in (1, \infty)$ be fixed. Using the above equality with $n = 1$ to the points $\sqrt{x}$ and $\sqrt{y}$ we get $\varphi(x) = \varphi(y)$, i.e. $\varphi_{|(1,\infty)}$ is constant. We know that φ is continuous at 1, therefore $\varphi_{|[0,1]} = \varphi(1) = \varphi_{|(1,\infty)}$. □

Remark 4.5.2. Note that for the maps $[0, \infty) \ni x \to x^2$ and $[0, \infty) \ni x \to \sqrt{x}$ we have four invariant compact sets, namely: $K_1 = \{0\}$, $K_2 = \{1\}$, $K_3 = \{0, 1\}$ and $K_4 = [0, 1]$. The below solutions show that we cannot use in the above propostion K_1 nor K_2 instead of K_3. Define $\phi_1, \phi_2 : [0, \infty) \to \mathbb{R}$ by

$$\phi_1(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \end{cases} \quad \text{and} \quad \phi_2(x) = \begin{cases} 1, & \text{if } x \in [0, 1) \cup (1, \infty), \\ 0, & \text{if } x = 1. \end{cases}$$

It is easy to check that ϕ_1 and ϕ_2 are solutions of equation (4.9) but they are discontinuous at 0 and 1, respectively. However, we can use K_4 instead of K_3 in the above proposition as $K_3 \subset K_4$. Nevertheless K_3 is the optimal choice.

As we have seen in the last example the continuity at each point of the invariant compact set have not to guarantee that the solution is constant if the functions $f_\omega, \omega \in \Omega$, are not contractions. Another example of a functional equation for the which inner functions are not contractions we will meet and solve in the next chapter.

Chapter 5

Appendix: Further discussion on the Derfel's problem

At the end of this thesis I would like to present my final thoughts and ideas connected with the original problem posed by Gregory Derfel, namely with equation (2.8). We saw an influence of invariant compact sets on solutions of linear functional equations in Chapter 4. We learnt that there is always a natural background in the functional equations theory, made by the Kuratowski equivalence relation – Chapter 3. Now we are reacher in that knowledge and we can go back to the analysis of equation (2.8) by this perspective and discover some new results. In the current chapter I present my attempts and thoughts which can help better emphasise the difficulties appearing in the study of equation (2.8).

5.1 How not to solve equation (2.8)

After three years of my attempts I realised that I had been trying to solve equation (2.8) on each equivalence class of the Kuratowski relation $\sim$ separately. More precisely, I always tried to prove that all solutions are constant on each equivalence class. During my work this restriction appears probably always but, in fact, sometimes it was hidden for me. Today I know that it was one of reasons of my failures. In my opinion the attempts of showing that all bounded continuous solutions of equation (2.8) are constant on equivalence classes of the relation $\sim$ seem to be a wrong attitude.

One can observe that in the case like this equvialence classes have a common accumulation point, namely 0. If every solution of equation (2.8) were constant on each equivalence class, then we could prove that every bounded solution of (2.8), which is continuous only at 0, must be constant. This regularity assumption seems to be too weak. We see that 0 is a fixed point of the function $\mathbb{R} \ni x \to -2x$ but the map $\mathbb{R} \ni x \to x-1$ has no finite fixed points. Therefore continuity at 0 is insufficient in my opinion. Moreover, we see that iterates of the inner functions are completly irregular in this case.

I would like to present a functional equation which is also a particular case of the archetypal equation, but as opposed to equation (2.8) we are able to solve it completely. This example demonstrates the above thoughts and the problems appearing in equation (2.8).

I decided to consider a functional equation of the form

$$\varphi(x) = \frac{1}{2}\varphi(-x-2) + \frac{1}{2}\varphi(2x+1) \tag{5.1}$$

for this purpose. This equation is obviously a particular case of the archetypal equation in which

$$K = \frac{1}{2}\ln|-1| + \frac{1}{2}\ln|2| = \frac{1}{2}\ln 2 > 0$$

and the rescaling parameter takes positive and negative values. One can note that the mappings $\mathbb{R} \ni x \to -x-2$ and $\mathbb{R} \ni x \to 2x+1$ have the same fixed point: -1. Therefore it is an example of the third degenerated case of the archetypal equation and we can apply Theorem 2.2.3 asserting that every bounded continuous solution of (5.1) is constant. We shall prove now something more, namely that every bounded solution of this equation, which is continuous only at -1, is constant. This is true since $\{-1\}$ is an invariant compact set.

I will present my solution of equation (5.1) step by step. At first we need to prove a technical lemma. Remind that a function $f : \mathbb{R} \to \mathbb{R}$ is called an *involution* if $f^2 = \mathrm{Id}_{\mathbb{R}}$.

Lemma 5.1.1. *Let $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ be commuting homeomorphisms and f_1 be an involution. If $\varphi : \mathbb{R} \to \mathbb{R}$ is a bounded solution of the equation*

$$\varphi(x) = \frac{1}{2}\varphi\big(f_1(x)\big) + \frac{1}{2}\varphi\big(f_2(x)\big), \tag{5.2}$$

then $\varphi_{|[a]_\sim}$ is constant for every $a \in \mathbb{R}$.

Proof. At the beginning we will prove that

$$\varphi(a) = \varphi\big((f_1^m \circ f_2^n)(a)\big) \quad \text{for every } m, n \in \mathbb{N} \cup \{0\} \text{ and } a \in \mathbb{R}. \tag{5.3}$$

Fix $a \in \mathbb{R}$ and put $\varepsilon = \varphi(a) - \varphi(f_1(a)) = \varphi(f_2(a)) - \varphi(a)$. Our goal is to show that $\varepsilon = 0$. For this purpose we shall prove inductively the following equalities

$$\varphi\big((f_1 \circ f_2^{n-1})(a)\big) = \varphi(a) - \frac{3^{n-1}+1}{2}\varepsilon \quad \text{for every } n \in \mathbb{N} \tag{5.4}$$

and

$$\varphi\big(f_2^n(a)\big) = \varphi(a) + \frac{3^n - 1}{2}\varepsilon \quad \text{for every } n \in \mathbb{N}. \tag{5.5}$$

First of all note that the above equalities hold for $n = 1$ because of the definition of ε. Suppose that they are true for every $n \in \{1, 2, ..., k\}$, where $k \in \mathbb{N}$ is fixed. We prove that (5.4) and (5.5) are satisfied for $k + 1$. Using the fact that φ is a solution of equation (5.2) and f_1 is an involtion we get

$$\begin{aligned}
\varphi\big((f_1 \circ f_2^{k-1})(a)\big) =& \frac{\varphi\big((f_1^2 \circ f_2^{k-1})(a)\big) + \varphi\big((f_2 \circ f_1 \circ f_2^{k-1})(a)\big)}{2} \\
=& \frac{\varphi\big(f_2^{k-1}(a)\big) + \varphi\big((f_1 \circ f_2^k)(a)\big)}{2}.
\end{aligned}$$

Hence

$$\begin{aligned}
\varphi\big((f_1 \circ f_2^k)(a)\big) =& 2\varphi\big((f_1 \circ f_2^{k-1})(a)\big) - \varphi\big(f_2^{k-1}(a)\big) \\
=& 2\left[\varphi(a) - \frac{3^{k-1}+1}{2}\varepsilon\right] - \left[\varphi(a) + \frac{3^{k-1}-1}{2}\varepsilon\right] \\
=& \varphi(a) - (3^{k-1}+1)\varepsilon - \frac{3^{k-1}-1}{2}\varepsilon \\
=& \varphi(a) - \frac{2 \cdot 3^{k-1} + 2 + 3^{k-1} - 1}{2}\varepsilon = \varphi(a) - \frac{3^k + 1}{2}\varepsilon.
\end{aligned}$$

Similarly

$$\varphi\big(f_2^k(a)\big) = \frac{\varphi\big((f_1 \circ f_2^k)(a)\big) + \varphi\big(f_2^{k+1}(a)\big)}{2},$$

and thus

$$\begin{aligned}
\varphi\big(f_2^{k+1}(a)\big) =& 2\varphi\big(f_2^k(a)\big) - \varphi\big((f_1 \circ f_2^k)(a)\big) \\
=& 2\left[\varphi(a) + \frac{3^k - 1}{2}\varepsilon\right] - \left[\varphi(a) - \frac{3^k+1}{2}\varepsilon\right] \\
=& \varphi(a) + \frac{2 \cdot 3^k - 2 + 3^k + 1}{2}\varepsilon = \varphi(a) + \frac{3^{k+1} - 1}{2}\varepsilon.
\end{aligned}$$

Therefore, in view of the mathematical induction, (5.4) and (5.5) are satisfied for every $n \in \mathbb{N}$. We see that the sequences $(\frac{3^{n-1}+1}{2})_{n\in\mathbb{N}}$ and $(\frac{3^n-1}{2})_{n\in\mathbb{N}}$ are unbounded. Since we assumed that φ is bounded, we have $\varepsilon = 0$. This fact, jointly with equalities (5.4) and (5.5), implies (5.3).

We are going to show that $\varphi_{|[a]_\sim}$ is constant. Take any $b \in [a]_\sim$. We have to prove that $\varphi(a) = \varphi(b)$. Since f_1 and f_2 are commuting we can find $n, m \in \mathbb{Z}$ such that

$$b = (f_1^n \circ f_2^m)(a). \tag{5.6}$$

Remind that f_1 is an involution, and thus this function satisfies the equality $f_1^n = f_1^{-n}$. Therefore we may assume without loss of generality that $n \geq 0$. Note that if $m \geq 0$, then $\varphi(a) = \varphi(b)$ because of equality (5.3). It remains to consider the case $m < 0$. Then equality (5.6) can be rewritten equivalently as

$$a = (f_1^{-n} \circ f_2^{-m})(b).$$

Putting the equality $f_1^n = f_1^{-n}$, we get $a = (f_1^n \circ f_2^{-m})(b)$. Now using equality (5.6) we obtain

$$\varphi(a) = \varphi\big((f_1^n \circ f_2^{-m})(b)\big) = \varphi(b)$$

and the proof has been completed. □

We will use the above lemma in solving equation (5.1).

Proposition 5.1.1. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a bounded solution of equation* (5.1). *If φ is continuous at -1, then φ is constant.*

Proof. Let $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ be given by the formulas

$$f_1(x) = -x - 2 \quad \text{and} \quad f_2(x) = 2x + 1.$$

One can check that $f_1 \circ f_2 = f_2 \circ f_1$ and f_1 is an involution. Moreover, we have

$$f_2^{-1}(x) = \frac{1}{2}x - \frac{1}{2} \quad \text{for every } x \in \mathbb{R}.$$

Lemma 5.1.1 asserts that $\varphi_{|[x]_\sim}$ is constant for every $x \in \mathbb{R}$. Since f_2^{-1} is a contraction, then the Banach contraction principle implies that -1 is an accumulation point of $[x]_\sim$ for all $x \in \mathbb{R}\setminus\{-1\}$. If $x = -1$, then $[-1]_\sim = \{-1\}$. Since φ is continuous at -1 and it is constant on all equivalence classes, we have $\varphi_{|[x]_\sim} = \varphi(-1)$ for each $x \in \mathbb{R}$. Hence φ is constant. □

The continuity at -1 is an essential assumption in the previous proposition.

Example 5.1.1. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a function given by the formula

$$\phi(x) = \begin{cases} 1, & \text{if } x \in \mathbb{R} \setminus \{-1\}, \\ 0, & \text{if } x = -1. \end{cases}$$

It is easy to check that ϕ is a solution of equation (5.1) and it has only one discontinuity point, namely -1.

We can solve equation (5.1) since iterates of the inner functions behave very regular and there exists a common fixed point of all the inner functions. In equation (2.8) we have not such possibilities – iterates are very chaotic and no invariant compact set exists.

Remark 5.1.1. *There is no invariant compact set for the inner maps appearing in equation* (2.8)*.*

Proof. Assume on the contrary that there exists a compact set $K \subset \mathbb{R}$ such that $-2K \subseteq K$ and $K - 1 \subseteq K$. At the beginning note that if K is a singleton, then the inlcusion $K - 1 \subseteq K$ cannot be satisfied. It remains to consider the case when K contains at least two elements. Then the diameter of the set $-2K$ is greater than the diameter of K. This observation implies that the inclusion $-2K \subseteq K$ cannot hold. □

5.2 Open problems

As we said in Chapter 2 we have very limited knowledge of non-degenerated case of the archetypal equation with positive K and $P(\alpha < 0) > 0$. There is no known such an example which we are able to solve completely in the class of bounded continuous functions. Therefore I tried to find some simpler equations to solve, in view of results from Chapters 2-4.

After the difficulties described in the previous section I would like to propose a particular case of the archetypal equation which seems to be easier to solve than equation (2.8). As we have seen in Proposition 4.5.1 and in Lemma 5.1.1 the equations for which iterates of inner functions behave more regularly are easier to solve. If at least one of the inner functions is an involution, then everything seems to be simpler.

Problem 1 Does there exist a non-constant bounded continuous solution $\varphi : \mathbb{R} \to \mathbb{R}$ of the equation

$$\varphi(x) = \frac{1}{2}\varphi(1-x) + \frac{1}{2}\varphi(2x)?$$

The above equation is also interesting since $K = \frac{1}{2}\ln 2 > 0$ and the rescaling parameter attains positive and negative values as well. But in the case like this the function $\mathbb{R} \ni x \to 1-x$ is an involution. Thus I suppose that it can help with its solution. Nevertheless, the problem is left open.

Problem 2 Is there any bounded and non-constant solution $\varphi : \mathbb{Z} \to \mathbb{R}$ of the Derfel equation (2.8)?

In that problem, without loss of generality we may replace $\mathbb{Z}$ by any countable set $E \subset \mathbb{R}$ fulfilling the conditions

$$E - 1 \subseteq E \quad \text{and} \quad -2E \subseteq E.$$

Problem 3 Describe all bounded continuous solutions $\varphi : \mathbb{R} \to \mathbb{R}$ of the equation

$$\varphi(x) = \frac{1}{2}\varphi(x-1) + \frac{1}{2}\varphi(2x).$$

Theorem 4.3 (a) from [10] asserts that if a solution φ has the limits $\lim_{x\to-\infty}\varphi(x)$ and $\lim_{x\to\infty}\varphi(x)$, then it must be a linear combination of the constant and the canonical (see Theorem 3.3 in [10] for details) solution. To the best of my knowledge there is no full description of bounded continuous solutions of the above equation without additional assumptions. If we omit the assumption of the existence of limits at the inifnites, then some new solutions may appear.

Problem 4 In Theorem 4.1.1 almost all functions $f_\omega, \omega \in \Omega$, are Matkowski type contractions. It seems to be possible to extend Theorem 4.1.1 to the case when the inner functions are Matkowski type contraction on sufficiently large subset of Ω (from the point of view of the measure P). It was made for the archetypal equation – Theorem 2.2.4. For the closest future I plan to focus my research in this direction.

Problem 5 In the literature there are not too many results guaranteeing the existence of invariant compact sets. It seems to be an interesting idea to get some new general theorems about existence of such sets since, as we have seen, they can be very useful in solving of functional equations.

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Streszczenie

Badania i rezultaty zamieszczone w rozprawie poświęcone są pewnym liniowym równaniom funkcyjnym nieskończonego rzędu. Punktem wyjścia do podjęcia przeze mnie badań w tym kierunku był problem postawiony przez Gregory'ego Derfla na *21st European Conference on Iteration Theory*, która została zorganizowana w 2016 roku w Innsbrucku (Austria). Swoje pytanie prof. Derfel powtórzył także rok później w trakcie *55th International Symposium on Functional Equations*, które miało miejsce w Chengdu, w Chinach. Gregory Derfel zapytał mianowicie o to, czy wszystkie ograniczone i ciągłe funkcje $\varphi : \mathbb{R} \to \mathbb{R}$ spełniające równanie

$$\varphi(x) = \frac{1}{2}\varphi(x-1) + \frac{1}{2}\varphi(-2x) \tag{1}$$

są stałe?

Sam problem jest związany z tzw. równaniem archetypicznym, którym zajmuje się prof. Derfel. Równanie archetypiczne jest to równanie funkcyjne postaci

$$\varphi(x) = \iint_{\mathbb{R}^2} \varphi\big(a(x-b)\big)\mu(da, db), \tag{2}$$

gdzie μ jest ustaloną borelowską miarą probabilistyczną określoną na płaszczyźnie $\mathbb{R}^2$, a $\varphi : \mathbb{R} \to \mathbb{R}$ szukaną funkcją. Wybierając konkretne miary μ otrzymujemy znane i badane wcześniej równania. Oznacza to, że analizując równanie (2) możemy badać jednocześnie obszerne klasy liniowych równań funkcyjnych. Ta szeroka perspektywa, którą w ten sposób osiągamy, uzasadnia trafność nazwy *równanie archetypiczne*. Warto dodać, że dla niektórych wyborów miary μ równanie (2) redukuje się do pewnych równań różniczkowych, np. równania pantografu.

Równanie (2) zostało dokładnie zbadane przez: L. Bogacheva, G.Derfla i S. Molchanova. Wiadomo między innymi, że ogromny wpływ na jego rozwiązania w klasie funkcji ograniczonych i ciągłych ma parametr K definiowany jako

poniższa całka

$$K := \iint_{\mathbb{R}^2} \ln|a|\mu(da, db).$$

Można wykazać, że dla $K < 0$ przy pewnych dodatkowych technicznych założeniach, równanie (2) nie ma w klasie funkcji ograniczonych i ciągłych innych rozwiązań niż funkcje stałe, w przypadku gdy $K > 0$ oraz $\mu((0, +\infty) \times \mathbb{R}) = 1$ potrafimy skonstruować niestałe rozwiązania. Sytuacja kiedy $K > 0$ i $\mu((-\infty, 0) \times \mathbb{R}) > 0$, a tak jest w równaniu (1), jest po dziś dzień niezbyt dobrze zbadana i pełna znaków zapytania. W rozprawie doktorskiej stawiam sobie za cel pogłębienie wiedzy o równaniu archetypicznym w tym przypadku.

Cała rozprawa została podzielona na pięć części. Rozdział 1 ma charakter wprowadzający i zawiera spis najważniejszych twierdzeń, faktów i oznaczeń stosowanych w kolejnych fragmentach rozprawy. Znajdują się tu dwie sekcje zatytułowane „Układy dynamiczne" oraz „Miara i całka".

W Rozdziale 2 badam rozwiązania równania (2) osiągające wartość najmniejszą bądź największą dla szerokiej klasy miar μ, w szczególności analizie poddane zostaje równanie (1). Główny mój wynik z tej części rozprawy implikuje między innymi, że jeżeli rozwiązanie ograniczone i ciągłe $\varphi : \mathbb{R} \to \mathbb{R}$ równania (1) osiąga ekstremum globalne, to φ musi być funkcją stałą. Poza tym stałość ograniczonych i ciągłych rozwiązań równania Derfla uzyskałem także przy założeniu, że istnieje choć jedna z granic $\lim_{x\to-\infty} \varphi(x)$ lub $\lim_{x\to+\infty} \varphi(x)$. Dodatkowo udało mi się zbadać jaki kształt powinno mieć potencjalne niestałe rozwiązanie φ równania (1) – okazało się, że taka funkcja musi oscylować zarówno w $-\infty$ jak i w $+\infty$; dokładniej: kiedy kierujemy się w stronę $-\infty$ lub $+\infty$, to wykres rozwiązania powinien nieskończenie wiele razy zbliżyć się nieskończenie blisko do kresu dolnego jak i do kresu górnego zbioru wartości rozwiązania φ.

Rozdział 3 zawiera pewne uogólnienie moich głównych wyników z Rozdziału 2 na równania funkcyjne postaci

$$\varphi(x) = \sum_{i\in I} p_i\varphi\big(f_i(x)\big), \tag{3}$$

gdzie $I \subseteq \mathbb{Z}$, dla każdego $i \in I$ funkcja $f_i : \mathbb{R} \to \mathbb{R}$ jest homeomorfizmem i $p_i \in (0, 1)$, przy czym $\sum_{i\in I} p_i = 1$. W ogólności dane homeomorfizmy nie muszą być funkcjami afinicznymi, gdyż w przeciwnym razie trafiamy ponownie do Rozdziału 2, ponieważ równanie (3) może zostać potraktowane jako szczególny przypadek równania archetypicznego. W Rozdziale 3 zastosowanie ma relacja równoważności generowana przez funkcje z rodziny $\{f_i\}_{i\in I}$, która została opisana po raz pierwszy przez Kazimierza Kuratowskiego. Wykazałem, że pod pewnymi warunkami – wprowadzonymi przeze mnie warunkami zgodności – każde ograniczone rozwiązanie równania (3), które osiąga na klasie abstrakcji wyznaczonej przez relację Kuratowskiego ekstremum globalne, musi być stałe na tej klasie abstrakcji. To ogólne twierdzenie można naturalnie zastosować do badania rozwiązań ciągłych, gdy wspomniane klasy abstrakcji są zbiorami gęstymi – daje to w efekcie stałość rozwiązania na całym zbiorze $\mathbb{R}$.

Rozdział 4 jest poświęcony najogólniejszemu równaniu funkcyjnemu pojawiającemu się w całej rozprawie. W przeciwieństwie do Rozdziałów 2 i 3 nie ograniczamy się do funkcji, które przekształcają prostą $\mathbb{R}$ w siebie, ale zupełną przestrzeń metryczną (X, d) w ośrodkową przestrzeń Banacha $(Y, \|\cdot\|)$. Mamy dodatkowo daną przestrzeń probabilistyczną $(\Omega, \mathcal{A}, P)$ i przekształcenie $f : \Omega \times X \to X$, które jest $\mathcal{A}$-mierzalne przy każdej ustalonej wartości $x \in X$. W Rozdziale 4 badamy ograniczone i borelowskie rozwiązania $\varphi : X \to Y$ równania

$$\varphi(x) = \int_\Omega \varphi\big(f(\omega, x)\big) P(d\omega), \tag{4}$$

gdzie do całkowania funkcji o wartościach wektorowych używamy całki Bochnera. Głównym przedmiotem zainteresowań tej części rozprawy są *zwarte zbiory niezmiennicze*, tzn. zwarte podzbiory K przestrzeni X, które dla wszystkich $\omega \in \Omega$ spełniają warunek

$$f(\omega, K) \subseteq K.$$

Okazuje się często, że ciągłość funkcji φ w każdym punkcie takiego zbioru implikuje jej stałość. W Rozdziale 4 dowodzę również twierdzenia gwarantującego istnienie zwartych zbiorów niezmienniczych dla pewnych rodzin kontrakcji.

Ostatni rozdział rozprawy, pełniący rolę podsumowania lub dodatku, zawiera dyskusję równania (1). Wyjaśniam w nim, dlaczego rozwiązanie równania (1) w klasie funkcji ograniczonych i ciągłych, bez żadnych dodatkowych założeń nałożonych na funkcję φ, jest tak trudnym zadaniem. Wynika to z faktu, że w tym przypadku nie istnieje zwarty zbiór niezmienniczy. Ponadto iteraty funkcji wewnętrznych zachowują się w sposób całkowicie nieregularny, co nie ułatwia rozwiązania równania (1). Ostatnia część Rozdziału 5 poświęcona jest przedstawieniu problemów otwartych, które pojawiły się w trakcie moich badań.