Referee report on the Doctoral Dissretation

On continuous solutions of linear functional equations of infinite order

by Mariusz Sudzik

The PhD thesis of Mr. Mariusz Sudzik is devoted to linear functional equations of infinite order in a single variable.

Such equations may appear in the form of

a) integral-functional equations, or in the form of

b) functional-differential equations.

One of the most important instances of a) is the *archetypal equation*, and a celebrated representative of b) is the *pantograph equation*.

Such equations arise in a number of applications, ranging from economics to biology, to astrophysics, etc., and their bounded continuous (b.c.) solutions are of main importance there.

The thesis is divided into five chapters.

The first chapter contains some auxiliary facts from measure theory and dynamical systems. In particular, notion of Kuratowski equivalence relation, and Matkowski's generalized contraction principle are given there.

The main results are presented in Chapters 2 through 5.

Chapter 2 is devoted to the archetypal equation and based on the author's papers [36] and [37].

For this important class of equations Mr. Sudzik obtaines an interesting new result, which roughly speaking says that every b.c.-solution , which attains its absolute maximum or minimum, is constant. To prove this result, he develops an elegant elementary method, which I believe may be applied even in a more general situation.

In Chapter 3 the author extends and generalizes the main results of the previous chapter on functional equations with nonlinear transformations of the arguments For this purpose he introduces here a concept of *compatibility conditions*, and under this additional assumption proves results similar to those of Chapter 2.

Chapter 4 is devoted to general linear iterative equations and their invariant compact sets. Equations of this type were studied in the Polish school of functional equations for years. The most important contributions in this field have been made, in particular, by K. Baron, R. Kapica, J. Morawiec, J. Jarczyk, and W. Jarczyk.

In the last chapter the author considers an interesting example of the archetypal equation of *supercritical type* with two rescaling factors of opposite signs. Also, a list of open problems is presented here.

The thesis is a clearly and comprehensively written document.

I have a few remarks, though.

General

Linear iterative equations, and, in particular, archetypal equations, have numerous important applications. Some of them are mentioned incidentally on page 8, (ref. [32], [33], [35]).

I think this information should be expanded.

In particular, a special case of the archetypal equation is the so-called *two-scaled difference equation* or *refinement equation*. This equation plays a crucial role in wavelet theory [1,2] and also in subdivision schemes and curve design [3], which is a rapidly growing branch of approximation theory.

1. Daubechies, I. *Ten Lectures on Wavelets*. CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 61. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992.

2. Daubechies, I. and Lagarias, J.C. Two-scale difference equations I. Existence and global regularity of solutions. *SIAM J. Math. Anal.* **22** (1991),

3. Derfel, G., Dyn, N. and Levin, D. Generalized refinement equations and subdivision processes. *J. Approx. Theory* **80** (1995), 272–297.

Details

1. Preface, l.1: infinite instead of inifnite.
2. Introduction, p.7, l.-1: It should be oscillating instead of oscilating.
3. Introduction, p.8, l.5: It should be considerations istead of cosniderations.
4. Chapter 1, p.10, l.-8: appears, instead of appear.
5. Chapter 3, p. 51 l.-7: discussion, instead of dicussion.
6. Chapter 3, p. 52, l.-8: their behaviour, instead of them behaviour.
7. Chapter 4, p. 63, l. 4: analysis, instead of analyse.
8. Chapter 4, p. 63, l. 8: present, instead of presented
9. Chapter 4, p. 74, l. 12: phenomenon, instead of phenomena.

The results obtained by Mariusz Sudzik are interesting and new, and make an original contribution to the theory of functional equations.

The minor remarks of editorial order, listed above, do no affect on my high estimate of the thesis.

I am convinced that Mariusz Sudzik definitely deserves a PhD degree and recommend so without any reservations.

Prof. Gregory Derfel

10/08/21