Referee's report on Ph.D.-Thesis 'Quantum Correlations in PT-Symmetric Systems' written by Vinh Le Duc at the University of Zielona Gora.

The Ph.D.-Thesis are composed of four chapters, the first two are devoted to the introductions to classical and quantum correlations and PT-symmetric systems. The last two chapters then contain original results by V. Le Duc concerning the analysis of classical and quantum correlations of two- and three-mode systems endowed with PT-symmetry. At page 96, there occurs the list of 3 original papers by V. Le Duc reporting on the results in Ph.D.-Thesis as well as the list of four Duc's contributions at the conferences.

Quantification of the entanglement (quantum correlations) discussed in Chapter 1 is based on the commonly used concurrence and negativity in case of bipartite states taking into account both pure and mixed states. Generalization is obtained for systems composed of general N parts using reduced bipartitions. Special attention is devoted to a three-mode system for which the tripartite negativity is introduced. A great deal of attention is devoted to the steering which is a specific form of quantum correlations that allows to steer one subsystem by the measurement made on the remaining subsystem. The quantum correlations identified by the violation of the Bell inequalities are finally mentioned as they represent the strongest possible form of quantum correlations that lead to the most pronounced declinations of quantum system behavior with respect to its classical counterpart. I appreciate the formulation by nonclassicality inequalities that allow to treat at the same footing both the steering and the Bell-like quantum correlations. Also introduction to the second- and fourth-order correlation functions that give the basic tools for describing the statistical properties of fields is given.

In Chapter 2, parity and time symmetries are discussed in detail in relation to non-Hermitian Hamitonians with real eigenvalues. Orthogonality and normalization of eigenfunctions of non-Hermitian Hamiltonians are discussed. Among others, it is shown that a system with parity symmetry preserves normalization, which is not the case of time symmetric systems.

Chapters 3 and 4 contain the original Duc's scientific results. In Chapter 3, the second and fourth-order coherence of a two mode PT-symmetric system is analyzed considering two in certain sense limiting initial conditions: one photon in the damped mode and one photon in the amplified mode. It is demonstrated that the second- and fourth-order coherences behave in the complementary way. The system is analyzed for four different values of damping/amplification constants. The negativity and steering are analyzed to quantify quantum correlations in the analyzed system. It is shown that the greater the damping/amplification constant is, the less entangled the system is. Typical examples of the system time evolution are presented. However, the main results are summarized in the graphs showing the maximal available values of quantum-correlation parameters and their temporal asymptotic values. Chapter 4 with its analysis of two configurations of a thee mode system represents generalization of the model of Chapter 3 by putting an additional mode inbetween the damped and amplified modes. In the first configuration, only this 'bridge' is considered. The second configuration even adds a direct 'connection' between the damped and amplified modes. Three different initial conditions are considered. Entanglement is analyzed both determining bipartite negativities and the tripartite negativity. It is shown that the system has only little potential to generate the genuine tripartite entanglement quantified by the tripartite negativity. On the other side, there occur many ways for steering of a mode by another mode. It is shown in general that when a mode is initially excited its potential to steer another mode considerably increases.

When reading the Ph.D.-Thesis I have found some items that might be misleading and should be clarified:

1) In Eq. (1.8), indices are missing on the r.h.s.

2) In Eq. (1.15), N as the number of modes should be defined.

3) In Eq. (1.20), the upper indices should be put into parentheses.

4) The sentense at the end of page 19 'steerable state forms a special group of the states that violate Bell inequality' should be explained/corrected.

5) Above Eq. (1.102), it is written that 'When the time-delay τ->0, $G^{(2)}(\tau)$ tends to be zero'. Please, explain this statement.

6) When bunching is discussed in the paragraph below Eq. (1.103), the results valid for a single-mode thermal field are mentioned. The single-mode character of the field as an important parameter should be mentioned.

7) Above Eq. (2.12), is the form of Hamiltonian H written correctly?

8) Concerning the indices denoting the mode that steers and the mode that is steered, it looks like that, at different parts of Ph.D.-Thesis, these modes are interchanged.

9) In Figs. 4.27, 4.28, and 4.29, it should be mentioned in the captions, that the r.h.s. graphs are insets of those at the l.h.s.

I have the following questions related to the content of the Ph.D.-Thesis:

1) In Eq. (1.59), parameter Z_j depends on the used measurable operators. Could you discuss this dependence in detail?

2) In Eq. (1.82), temporal mean value is introduced, in relation to the second-order coherence function. However, in the developed models, ensemble averages are determined. Could you discuss the relation between these two types of averages?

3) In the literature devoted to coherence, degrees of coherence are frequently defined and used. Would it be possible to discuss them in relation to the quantities defined in Ph.D.-Thesis?

4) Liouvillian of a damped and an amplified mode is written in Eq. (3.9). How the form of the Liouvillian for the amplified mode is derived? What kind of reservoir is assumed in the derivation?

5) The graphs in Fig. 3.6 show that the minimal and maximal values of second- and fourth-order coherence functions depend qualitatively on the initial conditions. Why this is so? Is there some simple physical explanation?

6) The curves in Fig. 3.9 for the maximal value of negativity suddenly loose monotonic behavior as γ/β increases. Could it be a consequence of numerical errors?

7) The results of numerical calculations predict nonzero asymptotic values for the negativity (see Fig. 3.10). One could be surprised arguing that, as the time increases and the fluctuating reservoir forces contribute constantly with certain value of the noise, quantum correlations are gradually concealed by this noise. However, the results indicate different behavior. Why? Is there a simple physical explanation?

8) In my opinion, it would help in some cases to draw in parallel to the maximal/minimal values of parameters characterizing quantum correlations also time instants at which these values are reached.

9) In some graphs of Chapter 4 (e.g., Fig. 4.11), the curves valid for $\kappa=0$ and $\kappa=0.1\beta$ are close. It would be interesting to compare the situation also for κ considerably larger to see what is the influence of the channel directly connecting the damped and amplified modes.

10) How the asymptotic values of parameters were technically obtained?

11) Is it possible to predict how the solution would look like for initial coherent states?

12) What are some prospective geometric configurations/generalizations of the discussed models? Why?

In general, the Ph.D.-Thesis is well written in good English and relatively easy to follow. It contains many interesting results related to the quantum coherence of optical fields in two- and three-mode PT-symmetric systems. It reveals several interesting ways for the generation of entangled states and states allowing steering of modes. The results have already been published in 3 impacted scientific journals, which confirms their quality and importance. **I am pleased to recommend Ph.D.-Thesis by V. Le Duc for defense. After successful defense, I agree with awarding Vinh Le Duc a Ph.D. degree.**

In Olomouc, October 18, 2022

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