Summary of the PhD Thesis written by Mariusz Sudzik

The scientific research presented in this PhD thesis is devoted to linear functional equations of infinite order. The starting point of my work was a problem posed by Gregory Derfel during the *21st European Conference on Iteration Theory* held in Innsbruck (Austria) in 2016. He asked if there existed a non-constant bounded continuous solution $\varphi : \mathbb{R} \to \mathbb{R}$ of the equation

$$\varphi(x) = \frac{1}{2}\varphi(x-1) + \frac{1}{2}\varphi(-2x). \tag{1}$$

The question was repeated by him one year later on the *55th International Symposium on Functional Equations* in Chengdu (China).

The above problem has been formulated in a very simple way, equation (1) has an uncomplicated shape. We need no advanced mathematical notions to pose the above question – it can be well understood by students of the first year of technical studies or even students of high school with an extended mathematics programme. Answering to the presented question turned out to be very difficult, as often as it is the case with mathematical problems which can be formulated very easy. The reasons of this fact and a partial answer to the Derfel's question are presented in this dissertation.

The problem posed by Derfel is tightly connected with the so-called *archetypal equation*, i.e. a functional equation of the form

$$\varphi(x) = \iint_{\mathbb{R}^2} \varphi\big(a(x-b)\big)\mu(da, db), \tag{2}$$

where μ is a given Borel probability measure defined on $\mathbb{R}^2$ and $\varphi : \mathbb{R} \to \mathbb{R}$ is a bounded continuous function. As we know, if equation depends on a parameter, then its shape can be various for different values of this parameter. In the case like this a role of the parameter is played by the measure μ and it can change the form of equation (2) drastically. As we will see in this text, the archetypal equation may sometimes reduce even to differential equations.

Equation (2) is very well examined in the case $\mu\big((0,\infty) \times \mathbb{R}\big) = 1$. In the paper [8] from 1989 Gregory Derfel discovered that a behaviour of the set of bounded continuous solutions of the archetypal equation depends significantly on the value of the number

$$K := \iint_{\mathbb{R}^2} \ln|a|\mu(da, db).$$

He proved that, under some technical assumptions, the archetypal equation has not non-constant bounded continuous solutions provided $K < 0$. He constructed also a non-constant bounded continuous solution for $K > 0$ when $\mu\big((0,\infty)\times\mathbb{R}\big) = 1$ additionaly. The fact that the support of μ is contained in the right half-plane is crucial in his proof. The case when $K > 0$ and $\mu\big((-\infty,0)\times\mathbb{R}\big) > 0$ has not been explored very well today and equation (1) examplifies this situation. In particular, in the thesis I would like to examine this class of functional equations and the topics which can help to find their solutions.

The whole text is splitted into five chapters. The first chapter contains basic facts from different branches of mathematics, mainly measure theory and dynamical systems. In particular, in Chapter 1 we present notations used in this text and basic terms which will be applied in sequent parts of the dissertation. The theorems contained in this part are presented without proofs with one exception but references are always given.

Chapter 2 is devoted to the archetypal equation. At the beginning of this part we present some results coming from the papers [6] and [7] by L. Bogachev, G. Derfel and S. Molchanov. They can help to understand a motivation of Gregory Derfel to proposed equation (1). The further sections of this chapter are based on the articles [17] and [18] which concern solutions of the archetypal equation attaining the global extremum. In particular, in Section 2.3 we will prove that each bounded continuous solution of equation (2) which attains its global extremum must be constant in the case $\mu((-\infty,0)\times\mathbb{R}) > 0$ if the measure μ satisfies some additional technical assumptions (see details in Theorem 2.3.1 and Theorem 2.3.2). We add that Theorem 2.3.1 implies that every bounded continuous solution of equation (1) is constant if it attains the global extremum. Section 2.4 contains a discussion on properties of solutions which do not attain the global extremum. In this section we prove a theorem saying that if such a non-constant solution exists, then it must be oscilating at the inifnities. It is worth adding that the existence of non-constant solutions in the class of bounded continuous functions is still an open problem.

Chapter 3 is a consequence of attempts in generalizing the results from Sections 2.3 and 2.4 on linear functional equations for which we have non-affine transforms of arguments. More precisely, the cosniderations of Chapter 3 refers to functional equations of the form

$$\varphi(x) = \sum_{i\in I} p_i \varphi\big(f_i(x)\big), \tag{3}$$

where $I \subseteq \mathbb{Z}$ is fixed, $p_i \in (0,1)$ are summing up to 1 and $f_i : \mathbb{R} \to \mathbb{R}$ is a homeomorphism for every $i \in I$. Section 3.1 contains a conception of the *compatibility conditions* which improve some dynamical aspects of iterates of the functions $f_i, i \in I$. In Section 3.2 we examine equation (3) under the comptability conditions. In particular, in this part we show that for a family $\{f_i\}_{i\in I}$ fulfilling any compatibility condition every bounded continuous solution $\varphi : \mathbb{R} \to \mathbb{R}$ of equation (3) is constant provided φ attains the global extremum. Section 3.3 contains the asymptotical analysis of solutions in the case when $\{f_i\}_{i\in I}$ contains at least one decresaing function. An example demonstrating the results of this chapter appears at the end of this section.

Chapter 4 is devoted to the most general linear functional equation appearing in this text. In this part we have given a probability space $(\Omega, \mathcal{A}, P)$, a complete metric space (X, d), a separable Banach space $(Y, \|\cdot\|)$ and a function $f : \Omega \times X \to X$ such that $f(\cdot, x)$ is

$\mathcal{A}$-measurable for every fixed $x \in X$. There we are interested in bounded Borel solutions $\varphi : X \to Y$ of the equation

$$\varphi(x) = \int_\Omega \varphi\big(f(\omega, x)\big) P(d\omega), \tag{4}$$

where the above integral denotes the Bochner integral. Equations of this form were examined by several authors for years. For example, more than 100 years ago Wacław Sierpiński considered in [15] a particular case of the above equation to characterize Cantor functions. One can observe that the archetypal equation is a very particular case of (4). It is worth adding that equations with affine transforms of arguments play a prominent role in applications. The reader is refered to [13], [14] or [16] for instance. Nowadays many papers connected with equation (4) and its generalization are written by a group of Polish mathematicans, namely: K. Baron, R. Kapica and J. Morawiec. They studied the various types of the convergence of the inner functions. The paper [2], written by Karol Baron, examplified their methods very well. The articles [4] and [5] are connected with a more general equation than (4). In the mentioned papers K. Baron and J. Morawiec examined the Lipschitzian solutions. Furthermore, to get solutions of equation (4) they also applied regular Markov-Feller operators which map the space of all Borel probability measures into itself (see [12] for instance). J. Jarczyk and W. Jarczyk are also interested in equation (4). In the recently published article [10] they considered equation (4) when the inner functions are means. Moreover, W. Jarczyk studied intensively linear functional equations under commutativity conditions – see [11]. We announce that some commutativity conditions will be assume in whole Chapter 4. General surveys of functional and functional-differential equations are found in G. Derfel [9], K. Baron and W. Jarczyk [3] and K. Baron [1].

Chapter 4 is based principally on the article [19]. The main topic of this part are invariant comptact sets. We say that a set $K \subseteq X$ is invariant if

$$\bigcup_{x \in K} \{f(\omega, x)\} \subseteq K \quad \text{ for all } \omega \in \Omega.$$

In Chapter 4 we will see that an influence of these sets on solutions of linear functional equations is crucial. Section 4.1 contains the main result and its proof. In Section 4.2 we prove some theorems connected with the existence of invariant compact sets. In Theorem 4.1.1 we impose some technical assumptions, that is why in Section 4.3 we present a lot of particular cases of this theorem. Section 4.4 is devoted to the case when the order of equation (4) is finite. The last section contains some examples and applications of invariant compact sets to linear functional equations.

In the last chapter we are going back to equation (1). In Section 5.1 we prove, among others, that for this equation there are no non-empty invariant compact sets. This is one of the reasons why the equation proposed by Derfel is so hard to solve. The other reasons are disccused also therein. In the next, and last, part of this dissertation we pose some open problems.

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