Different variants of matroids coloring

The doctoral thesis in the field of mathematical sciences, written by MSc Zofia Miechowicz under the supervision of professor Jarosław Grytczuk

The main scientific goal of the doctoral thesis is to investigate different variants of matroids coloring. The thesis contains 5 chapters.

Chapter 1 contains the introduction to the matroid theory. It includes the brief historical review, the main definitions and examples of classes of matroids.

The subject od the **Chapter** 2 is *cooperative* coloring of matroids.

Let $M_1, M_2, \ldots, M_k$ be a collection of matroids on the same ground set E. A coloring $c : E \to \{1, 2, \ldots, k\}$ is called *cooperative* if for every color j, the set of elements in color j is independent in M_j. In this chapter we prove that such coloring always exists provided that for every matroid M_j we have $\chi(M_j) \leq k$. We derive this fact from a generalization of Seymour's list coloring theorem for matroids, which asserts that every k-colorable matroid is also k-list colorable. We also point on some consequences for the game-theoretic variants of cooperative coloring of matroids.

In **Chapter 3** we consider a higher order analog of the *arboricity* of a graph, denoted by $\mathrm{arb}_p(G)$, introduced recently by Nešetřil, Ossona de Mendez, and Zhu [4]. It is defined as the least number of colors needed to color the edges of a graph so that each cycle C gets at least $\min\{|C|, p+1\}$ colors (we call such coloring *p-acyclic*). So, $\mathrm{arb}_1(G)$ is the usual arboricity of G, while $\mathrm{arb}_2(G)$ can be seen as a relaxed version of the *acyclic chromatic index* of G. We introduce p-acyclic coloring of graph G as a coloring of the graphic matroid $M(G)$.

We obtain explicit upper bounds on $\mathrm{arb}_p(G)$ for some classes of graphs. We prove that $\mathrm{arb}_p(G) \leq p+1$ for every planar graph of girth at least 2^{p+1}. A similar result holds for graphs with arbitrary fixed genus. We also demonstrate that $\mathrm{arb}_2(G) \leq 5$ for outerplanar graphs, which is best possible. By using *entropy compression* argument we prove that $\mathrm{arb}_p(G) \leq (\Delta-1)p+1$ for graphs of maximum degree Δ and sufficiently large girth.

In **Chapter 4** we investigate $\mathrm{arb}_p(M)$ for some classes of matroids, i.e. the uniform and partition matroids and the Catalan matroids. We also present results concerning the duals of a graphic matroids introduced by Nešetřil, Ossona de Mendez, and Zhu [4].

In **Chapter 5** we define and investigate *the strong arboricity* of graphs and its generalziation to matroids. An edge coloring of a graph G is *acyclic* if no cycle is monochromatic. The *arboricity* of a graph G, denoted by $\mathrm{arb}(G)$, is the least number of colors needed for an acyclic coloring of G. A coloring of G is *strongly acyclic* if after contraction of any single edge it is still woody. In other words, not only any cycle in G is monochromatic but also any *broken cycle*, i.e., a simple path arising by deleting a single edge from the cycle. The least number of colors in a strongly woody coloring of G is denoted by $\zeta(G)$ and called the *strong arboricity* of G.

We prove that $\zeta(G) \leqslant \chi_a(G)$, where $\chi_a(G)$ is the *acyclic chromatic number* of G (the least number of colors in a proper vertex coloring without a 2-colored cycle). In particular, we get that $\zeta(G) \leqslant 5$ for planar graphs and $\zeta(G) \leqslant 4$ for otuterplanar graphs. We conjecture that $\zeta(G) \leqslant 4$ holds for all planar graphs. We also prove that $\zeta(G) \leqslant 4(\mathrm{arb}(G))^2$ holds for arbitrary graph G. A natural generalziation of strong arboricity to *matroids* is also discussed, with a special focus on cographic matroids.

Zofia Miechowicz

Spis literatury

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Zofia Miechowicz